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Input-output stability of interconnected stochastic systems

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Input-output stability of interconnected stochastic systems

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TABLE OF CONTENTS

1. INTRODUCTION AND BACKGROUND MATERIAL

1.1 Introduction

One of the most basic questions one can ask about a system is whether or not it is stable. In this chapter we review briefly what is meant by stability or instability, methods of determining stability or instability, and the ways in which this thesis contributes to the area of stability theory. We are primarily interested in systems where the elements (system parameters, inputs, or outputs) are not precisely known and in some sense may be thought of as random. In addition, since practical physical problems are often difficult to handle due to their shear size, we investigate large systems that may be thought of as a collection of smaller, more easily handled subsystems, whose outputs and inputs are mutually related through some interconnecting structure. Many papers have been published recently on the interconnected system concept, as will be seen below.

1.2 Background Material

The classical approach to system stability originated with the work of Liapunov, a 19th century Russian mathematician. To use Liapunov's techniques, the system to be studied must be described by differential equations in state space format. Stability is then defined in terms of how the norm of the state vector behaves in response to various initial conditions. In general, the system is not assumed to be driven by some external forcing function. The analysis technique is Liapunov's direct method, which involves finding a real-valued continuous function of the state variables, $v(x)$, which has certain definiteness properties and whose derivative, $\dot{v}(x)$, evaluated along the trajectory of a solution of the differential equation, has other definiteness properties. If such a function can be found (called a v-function) with the correct definiteness properties, Liapunov's direct method guarantees stability (asymptotic stability, instability, etc.) of the equilibrium in question (see Hahn [8] or La Salle and Lefschetz [13] for a discussion of Liapunov's direct method). Although it can be shown that if a system is stable an appropriate v-function must exist, there is no general method for constructing v-functions. Recent work by Michel [21, 22], among others [2, 7, 18, 36], has simplified the problem of finding a v-function for certain classes of interconnected systems. The approach is to deduce an overall v-function from a weighted sum of subsystem v-functions. Stability is then determined on the basis of this overall v-function and the parameters in the interconnecting structure.

Liapunov stability concepts have been extended to stochastic systems. The most widely studied stochastic systems, in this respect, are those that can be described by Ito differential equations (see Kushner [12] or Arnold [1]). In engineering terms, the Ito differential equation represents a system driven by white noise. The solution to such a differential equation is a random process. The stability requirements for the stochastic system are that the system must be stable, in the deterministic sense, with probability one. By choosing an appropriate v-function, we can deduce the stochastic equivalent of the deterministic stability theorems mentioned above (Liapunov's direct method). Recent work by Michel and Rasmussen [23, 24, 31] has extended these ideas to the interconnected system structure, similar to the deterministic case.

In relatively recent times another useful definition of system stability has been developed; this is referred to as input-output stability. Input-output stability, in addition to being intuitively appealing to the engineer, has some practical advantages over the Liapunov approach. Typical results for input-output stability analysis are constructive. That is, they involve a step-by-step procedure for analysis such that any system in the particular class under consideration may be tested without searching for something as elusive as a v-function "that works." Another advantage is that, typically, the information needed to test a system may be found experimentally (refer to the circle theorems mentioned below and the frequency-domain results presented in Chapters 3, 4, and 5).

Input-output stability concepts were primarily introduced into systems theory by I. W. Sandberg and G. Zames (working independently). In input-output stability theory, we consider systems with inputs as well as outputs. It is usually assumed that the input belongs to some normed linear space. For input-output stability we require the output to belong to a similar normed linear space (and hence have a finite norm). In the usual setting, the system is in feedback or closed-loop form. In the forward path there is a "plant relation" and in the feedback path there is a "feedback relation." Combining these two relations into one, the system may be thought of as one overall relation between system inputs and system outputs. This overall relation is referred to as the "closed loop relation." The problem is to deduce conditions on the plant relations and feedback relations that imply closed loop relation stability (or instability).

A key result in input-output stability analysis is the small gain theorem, which states, roughly, that if the product of the plant relation gain and the feedback relation gain is less than unity, then the closed loop relation is stable. The power of this simple result is only fully realized in special cases. For example, when the forward path relation is a linear, time invariant causal convolution operator and the feedback path relation is a memoryless nonlinearity, Sandberg [32-34] and Zames [44, 45], for example, have obtained a generalization of the Nyquist stability criterion, which is referred to as the circle theorem.

Porter and Michel [30], Lasley and Michel [14, 15], and Miller and Michel [26, 27] demonstrated that input-output stability concepts are adaptable to large scale systems. These results show that the stability of certain systems may be determined graphically in circle theorem — like results and in results similar to the Popov criterion (for a discussion of the Popov stability criterion see, for example, Hahn [8]).

The application of input-output stability methods to stochastic systems is still in its infancy. This presents several difficulties. For instance, there is no common agreement as to what type of underlying linear space is most applicable for stochastic system stability (there are many from which to choose). In this paper we use three different sets of spaces and norms, each of which has been studied to some extent by a previous author (for the definitions of these types of stochastic stability consult Definitions 3.8, 3.9, 3.10, 4.1, and 5.1).

The basic work of Sandberg and Zames, formulated in terms of relations on linear spaces, is general enough, in principle, to be used with stochastic systems, however direct application of these basic results to a particular system is quite difficult. The circle criterion, for instance, was developed as a convenient method of applying these basic results to a (somewhat) restricted class of deterministic systems. Currently, a circle theorem for stochastic systems is being sought. The recent work of Willems and Blankenship [40] is a beginning in this direction. Blankenship, in his thesis [4], developed circle conditions for a class of stochastic systems, however they appear to be somewhat limited from either the control or the interconnected system standpoint because he requires that the system input be stochastically independent of past values of the system output. The circle conditions of Willems and Blankenship [40] suffer from the same restrictions. It is in this area that we begin our study. We relax the above restrictions on input and output independence and establish stability results for the interconnection of several types of subsystems in Chapter 3. Conditions placed on the subsystems for system stability may often be determined graphically. Also, in Chapter 3, we establish new instability results for certain classes of interconnected subsystems. In Chapter 4 the system is complicated by adding a nonlinearity. Single loop stability results are established for these systems. Chapter 5 contains stability results for a wide class of interconnected systems. In this chapter we make a more direct application of the circle theorem and Popov's theorem to the systems under study. Consequently, the results of Chapter 5 are frequency-domain results. Also, in this chapter, stochastic integral equation results are applied to systems that can be described by stochastic differential equations. Chapter 6 contains examples that use the results of Chapters 3, 4, and 5. The proofs of the theorems appear in the Appendices.

2. MATHEMATICAL NOTATION AND PRELIMINARIES

2.1 Notation

Let $A = [a_{ij}]$ denote an $n \times m$ matrix and let A^T denote the transpose of A. Let A^* denote the complex conjugate transpose of A. The inverse of a nonsingular $n \times n$ matrix, A, is denoted by A^{-1}. If C and D are real $n \times m$ matrices, then $C \geq D$ means $c_{ij} \geq d_{ij}$ for all i and j and $C \geq 0$ means $c_{ij} \geq 0$ for all i and j. Let I denote the $N \times N$ identity matrix. Let $\Lambda[M]$ denote the positive square root of the largest eigenvalue of M^*M. If the elements of a real matrix, B, depend on a real parameter, t, we say that B(t) is bounded if there exists a real number, M, such $|b_{ij}(t)| \leq M < \infty$ for all allowable t and all i and j. We define $R = (-\infty, \infty)$, $R^N = R \times R \times \ldots \times R$ (N times) and $R^+ = [0, \infty)$. If $x = [x_1, x_2, \ldots x_N]^T$ with $x_i \in R$ $(x \in R^N)$, then $|x| = (|x_1|^2 + |x_2|^2 + \ldots |x_N|^2)^{1/2}$. We will define I^+ by $I^+ = \{0, 1, 2, \ldots\}$.

The set of all real, Lebesgue-measurable N-vector-valued functions of the real variable, $t \in R^+$ is denoted by $H_{(N)}(R^+)$; and $L_{p(N)}(R^+) = \left\{ f \in H_{(N)}(R^+) : \int_0^\infty |f(t)|^p dt < \infty \right\}$, $1 \leq p < \infty$. If N = 1, we often write $L_p(R^+)$ instead of $L_{p(1)}(R^+)$. The inner product of two elements, f and g, of $L_{2(N)}(R^+)$ is denoted by

$$\langle f, g \rangle = \int_0^\infty f^T g dt.$$

The norm of $f \in L_{2(N)}(R^+)$ is defined by $||f|| = \langle f, f \rangle^{1/2}$. If $x \in H_{(N)}(R^+)$ we define the truncation of x by

$$x_T(t) = \begin{cases} x(t), & 0 \le t \le T \\ 0 \quad , & t > T \end{cases} \qquad t,\ T \epsilon R^+,$$

and the truncation operator, π_T, by

$$\pi_T x(t) = x_T(t), \qquad t,\ T \epsilon R^+.$$

The extended $L_{p(N)}(R^+)$ space, $L_{pe(N)}(R^+)$, is defined by

$$L_{pe(N)}(R^+) = \left\{ x \epsilon H_{(N)}(R^+):\ \ x_T \epsilon L_{p(N)}(R^+) \text{ for all } T \epsilon R^+ \right\}.$$

If H is an operator on $L_{pe}(R^+)$, we say H is causal if

$$\pi_T Hx(t) = \pi_T Hx_T(t) \qquad t,\ T \epsilon R^+,\ x \epsilon L_{2e}(R^+).$$

Let $A(t) = [a_{ij}(t)]$ be an arbitrary $N_1 \times N_2$ matrix-valued Lebesgue-measurable function of $t \epsilon R^+$. We say $A \epsilon K_{p(N_1 \times N_2)}(R^+)$, $1 \le p < \infty$, if $\int_0^\infty |a_{ij}(t)|^p dt < \infty$ for all i and j. If H is a convolution operator on $L_{2e(N)}(R^+)$, that is,

$$Hx(t) = \int_0^t h(t - \tau) x(\tau) d\tau$$

with $h \epsilon K_{1(N \times N)}(R^+)$, $x \epsilon L_{2e(N)}(R^+)$, then $\tilde{h}(s)$ will denote the Laplace transform of h(t),

$$\tilde{h}(s) = \int_0^\infty h(t) e^{-st} dt.$$

We refer to h(t) as a convolution kernel.

Definition 2.1. A convolution kernel, $h \epsilon K_{1(1 \times 1)}$ as specified above is said to possess Property L if

$$\inf_{\mathrm{Re}(s)\geq 0} |1 + \tilde{h}(s)| > 0.$$

Definition 2.2. Given a convolution kernel, h, as described above, we formally define the resolvent associated with h(t) as the real function $\bar{r}(t)$ that satisfies

$$\bar{r}(t - s) = h(t - s) - \int_s^t \bar{r}(t - \tau)\cdot h(\tau - s)ds \qquad (2.1)$$

As a result of the well-known Paley-Wiener theorem (see Miller [25]) if $h\epsilon K_{1(1\times 1)}(R^+)$ and h possess Property L, then $\bar{r}(t)$ exists, $\bar{r}\epsilon L_1(R^+)$ and $\bar{r}(t)$ satisfies Eq. 2.1.

Given a probability space, (Ω, F, P), denote by $X_{(N)}(\Omega)$ the space of N-dimensional real-valued random vectors over Ω which have finite second moments, that is, if $x(\omega) = [x_1(\omega), \ldots, x_N(\omega)]^T \epsilon X_N(\Omega)$, then $x_i(\omega)$ is F-measurable for $i = 1, 2, \ldots, N$, and $\int_\Omega x^T(\omega)x(\omega)dP(\omega) < \infty$. Let $H_{(N)}(R^+, \Omega)$ denote the space of all real, N-dimensional random processes over $R^+ \times \Omega$ such that if $x\epsilon H_{(N)}(R^+, \Omega)$, then $x(\cdot, \omega)\epsilon H_{(N)}(R^+)$ for fixed $\omega\epsilon\Omega$, and $x(t, \cdot)\epsilon X_{(N)}(\Omega)$ (for fixed $x\epsilon R^+$). Let S_∞ denote the set of all scalar, real-valued random processes, $x(t, \omega)$, on $R^+ \times \Omega$ such that

$$\sup_{t\epsilon R^+} Ex^2(t, \omega) < \infty.$$

Let $S_{\infty e}$ be defined as the set of all real-valued scalar random processes, $x(t, \omega)$, on $R^+ \times \Omega$ such that

$$\sup_{0\leq t\leq T} Ex^2(t, \omega) < \infty \quad \text{for every } T\epsilon R^+.$$

Analogously, let s_∞ be the set of all real-valued scalar random processes, $x(n, \omega)$, on $I^+ \times \Omega$ such that

$$\sup_{n \in I^+} Ex^2(n, \omega) < \infty,$$

and define $s_{\infty e}$ as the set of all real-valued scalar random processes, $x(n, \omega)$, on $I^+ \times \Omega$ such that

$$\sup_{0 \leq n \leq N} Ex^2(n, \omega) < \infty \qquad \text{for every } N \in I^+.$$

Denote by $L_{1(N)}(R^+, L_\infty(\Omega))$ the set of all real N-vector-valued random processes, $x \in H_N(R^+, \Omega)$, such that

$$\operatorname*{ess\,sup}_{\omega \in \Omega} \int_0^\infty |x(t, \omega)| dt < \infty.$$

Denote by $L_{2(N)}(R^+, L_\infty(\Omega))$ the set of all real N-vector-valued random processes, $x \in H_N(R^+, \Omega)$, such that

$$\operatorname*{ess\,sup}_{\omega \in \Omega} \int_0^\infty x^T(t, \omega) x(t, \omega) dt < \infty.$$

Let $A(t, \omega) = [a_{ij}(t, \omega)]$ be an arbitrary $N_1 \times N_2$ - matrix-valued random process with $a_{ij} \in H_{(1)}(R^+, \Omega)$. We say that $A \in K_{p(N_1 \times N_2)}(R^+, L_\infty(\Omega))$, $1 \leq p < \infty$ if

$$\operatorname*{ess\,sup}_{\omega \in \Omega} \int_0^\infty |a_{ij}(t, \omega)|^p dt < \infty \qquad \text{for all i and j.}$$

If $T \in R^+$, we define the truncation of $x(t, \omega)$ by

$$x_T(t, \omega) = \begin{cases} x(t, \omega) & \text{for } 0 \le t \le T \\ 0 & \text{for } t > T \end{cases}, \quad t, T \epsilon R^+,$$

and we define the extended space $E_{p(N)}$, p = 1 or 2, by

$$E_{p(N)} = \left\{ x \epsilon H_N(R^+, \Omega): \; x_T \epsilon L_{p(N)}(R^+, L_\infty(\Omega)), \; T \epsilon R^+ \right\}.$$

As in the deterministic case, π_T denotes the truncation operator. Let $E_{s(N)}$ denote those processes in $E_{2(N)}$ with time-derivatives in $E_{2(N)}$.

Definition 2.3. Let $\eta_{(N)}$ denote the collection of memoryless nonlinearities of the type

$$\psi(x(t, \omega), t, \omega) = [\psi_1(x_1(t, \omega), t, \omega) \ldots, \psi_N(x_N(t, \omega), t, \omega)]^T$$

for $x \epsilon H_N(R^+, \Omega)$, $t \epsilon R^+$, $\omega \epsilon \Omega$, where $\psi_i(g, t, \omega)$, i = 1, ..., N, are real-valued scalar functions of the real variables $g \epsilon R$ and $t \epsilon R^+$ and the variable $\omega \epsilon \Omega$ such that

(i) $P\left\{\omega: \psi_i(0, t, \omega) = 0, \; t \epsilon R^+, \quad i = 1, 2, \ldots, N\right\} = 1;$

(ii) there exist real numbers a and b such that

$$P\left\{\omega: a \le \frac{\psi(g, t, \omega)}{g} \le b, \; t \epsilon R^+, \; g \ne 0, \; i = 1, 2, \ldots, N\right\} = 1;$$

(iii) $\psi_i(x(t, \omega), t, \omega)$ is a Lebesgue-measurable function of t and an F-measurable function of ω whenever x is a Lebesgue-measurable function of t and an F-measurable function of ω; and

(iv) $\psi_i(x(t, \omega), t, \omega) \epsilon H_N(R^+, \Omega)$ whenever $x \epsilon H_N(R^+, \Omega)$.

Definition 2.4. The stochastic N × N matrix $A(\omega) = [a_{ij}(\omega)]$, with $a_{ij} \epsilon X_{(1)}(\Omega)$, is said to be stochastically stable if, for some positive, real γ,

$$P\left\{\omega\colon \operatorname{Re}(\lambda_k(\omega)) < -\gamma,\ k = 1, 2, \ldots, N\right\} = 1,$$

where $\lambda_k(\omega)$, $k = 1, 2, \ldots, N$ are the eigenvalues of $A(\omega)$.

2.2 Preliminaries

In this section we present two lemmas which are used throughout this thesis. The first deals with Minkowski matrices, also referred to as M-matrices (see Ostrowski [29] and Fiedler and Ptak [6]).

Definition 2.5. A square matrix, $A = [a_{ij}]$, is said to be an M-matrix if the off-diagonal elements are all nonpositive ($a_{ij} \leq 0$, $i \neq j$) and the principal minors are all positive.

Lemma 2.1. Let $A = [a_{ij}]$ be an $n \times n$ M-matrix. Then the matrix A is nonsingular and $A^{-1} > 0$.

Lemma 2.1 is proved in Ostrowski [29] and Fiedler and Ptak [6].

The second lemma concerns the asymptotic behavior of the function f(t) defined by

$$f(t) = \int_0^t k(t - \tau)h(\tau)d\tau, \tag{2.2}$$

where $k \epsilon L_1(R^+)$ and $h \epsilon L_2(R^+)$. The following lemma and the associated proof are presented informally in Sandberg [32].

Lemma 2.2. If in Eq. 2.2, $k \epsilon L_1(R^+) \cap L_2(R^+)$, then $f(t) \to 0$ as $t \to \infty$.

3. SECOND ORDER STOCHASTIC INPUT-OUTPUT STABILITY

3.1 Introduction

The first type of system to be investigated in this thesis involves control systems with gain terms modeled by stochastic processes. Recently, much work has been done in this area (see, for instance, Blankenship [4], Kleinman [10], Martin and Johnson [17], Willems and Blankenship [40], Willsky, et al. [41], and Wonham [43]). Such random gain terms occur in circuit models (see, for example, Bertram and Sarachik [3]), models of the human controller (see Levison, et al. [16]), models of round-off error in floating point arithmetic (see Blankenship [4]), and in other areas where the magnitude of the error associated with a signal is directly proportional to the signal magnitude (see Kleinman [10]).

A convenient mathematical starting point is to model the gain terms as multiplicative white noise (as in Willems and Blankenship [40]). This is the approach taken in the present chapter as well as in the following one. Initially the white noise gain term will be assumed to have zero mean. Results for this case will then be extended to the nonzero mean white noise case.

As in [40], we consider input-output stability and instability defined in terms of second moments. An extensive review of the various definitions of stochastic stability may be found in the survey paper by Kozin [11]; however, such a review is not particularly germane to this discussion and will not be dealt with here.

The emphasis of this chapter is on continuous-time systems; however, discrete-time systems are also treated (in stability Theorem 3.2 and in instability Theorem 3.6).

The necessary background material is presented in the next section, the main stability results are presented in Section 3.3, while in Section 3.4 corresponding instability results are given. All results of this chapter are proved in Appendix A.

3.2 Mathematical Background and Definitions

In this chapter we define the symbols $||\cdot||$, $||\cdot||_T$, and $||\cdot||_N$ as follows:

$$||x(t,\omega)|| = \sup_{t\in R^+} [Ex^2(t,\omega)]^{1/2} \quad \text{for } x\in S_\infty,$$

$$||x(t,\omega)||_T = \sup_{0\leq t\leq T} [Ex^2(t,\omega)]^{1/2} \text{ for } x\in S_{\infty e},\ T\in R^+,$$

$$||x(n,\omega)|| = \sup_{n\in I^+} [Ex^2(n,\omega)]^{1/2} \quad \text{for } x\in s_\infty, \text{ and}$$

$$||x(n,\omega)||_N = \sup_{0\leq n\leq N} [Ex^2(n,\omega)]^{1/2} \text{ for } x\in s_{\infty e},\ N\in I^+.$$

A similar convention is used for operator norms on these spaces. We consider continuous-time systems that may be modeled by the following set of functional equations:

$$\left.\begin{aligned} e_i(t, \omega) &= u_i(t, \omega) - C_i y_i(t, \omega) \\ y_i(t, \omega) &= H_i e_i(t, \omega) \\ u_i(t, \omega) &= r_i(t, \omega) + \sum_{j=1}^{m} B_{ij} y_j(t, \omega) \end{aligned}\right\} , \tag{3.1}$$

with $t \epsilon R^+$, and $i, j \epsilon M = \{1, 2, \ldots m\}$. For each $i, j \epsilon M$, r_i, e_i, y_i, and u_i are assumed to belong to $S_{\infty e}$; H_i is a relation on $S_{\infty e}$; C_i is assumed to be a stochastic operator on $S_{\infty e}$ such that

$$\|C_i y_i(t, \omega)\|_T \leq g_i \|y_i(t, \omega)\|_T , \; y_i \epsilon S_{\infty e}, \; T, \; g_i \epsilon R^+;$$

and B_{ij} is assumed to be a stochastic operator on $S_{\infty e}$ such that

$$\|B_{ij} y_j(t, \omega)\|_T \leq d_{ij} \|y_j(t, \omega)\|_T, \; y_j \epsilon S_{\infty e}, \; T, \; d_{ij} \epsilon R^+.$$

Here r_i is an input, e_i is an error, y_i is an output and u_i is an intermediate variable. System 3.1 may be viewed as an interconnection of m free or isolated subsystems, each described by equations of the form

$$\left.\begin{aligned} e_i(t, \omega) &= r_i(t, \omega) - C_i y_i(t, \omega) \\ y_i(t, \omega) &= H_i e_i(t, \omega), \quad i \epsilon M, \; t \epsilon R^+ \end{aligned}\right\} . \tag{3.2}$$

For the discrete-time case, we consider systems described by the set of functional equations

$$\left.\begin{aligned} e_i(n, \omega) &= u_i(n, \omega) - C_i y_i(n, \omega) \\ y_i(n, \omega) &= H_i e_i(n, \omega) \\ u_i(n, \omega) &= r_i(n, \omega) + \sum_{j=1}^{m} B_{ij} y_j(n, \omega) \end{aligned}\right\} \tag{3.3}$$

for $i, j \in M$, $n \in I^+$, where e_i, u_i, y_i, r_i, C_i, H_i, and B_{ij} are defined as in Eq. 3.1, with t replaced by n, T replaced by N, $S_{\infty e}$ replaced by $s_{\infty e}$, and $\|\cdot\|_T$ replaced by $\|\cdot\|_N$. Once again, System 3.3 may be viewed as an interconnection of m free or isolated subsystems, each described by equations of the form

$$\left.\begin{aligned} e_i(n, \omega) &= r_i(n, \omega) - C_i y_i(n, \omega) \\ y_i(n, \omega) &= H_i e_i(n, \omega) \end{aligned}\right\} . \tag{3.4}$$

We allow the relation H_i in Eqs. 3.1 through 3.4 to take on several different forms. It is the H_i that determines the "plant" or forward path characteristic of each loop. Figure 3.1 depicts System 3.1 or 3.3.

<u>Definition 3.1</u>. Continuous-time Subsystem 3.2 is said to be of <u>Type 1</u> if (informally)

$$y_i(t, \omega) = \int_0^t w_i(t, s) e_i(s, \omega) f_i(s, \omega) ds \tag{3.5}$$

with $t \in R^+$, $i \in M$, where $w_i(t, s)$ is a real nonanticipatory integral operator kernel, independent of ω, and f_i is a white noise process with

$$Ef_i(t, \omega) = 0 \qquad \text{for } t \in R^+, \text{ and}$$

$$E\left\{f_i(t, \omega) f_i(t + \tau, \omega)\right\} = \sigma^2(t)\delta(\tau), \ t, \ \tau \in R^+,$$

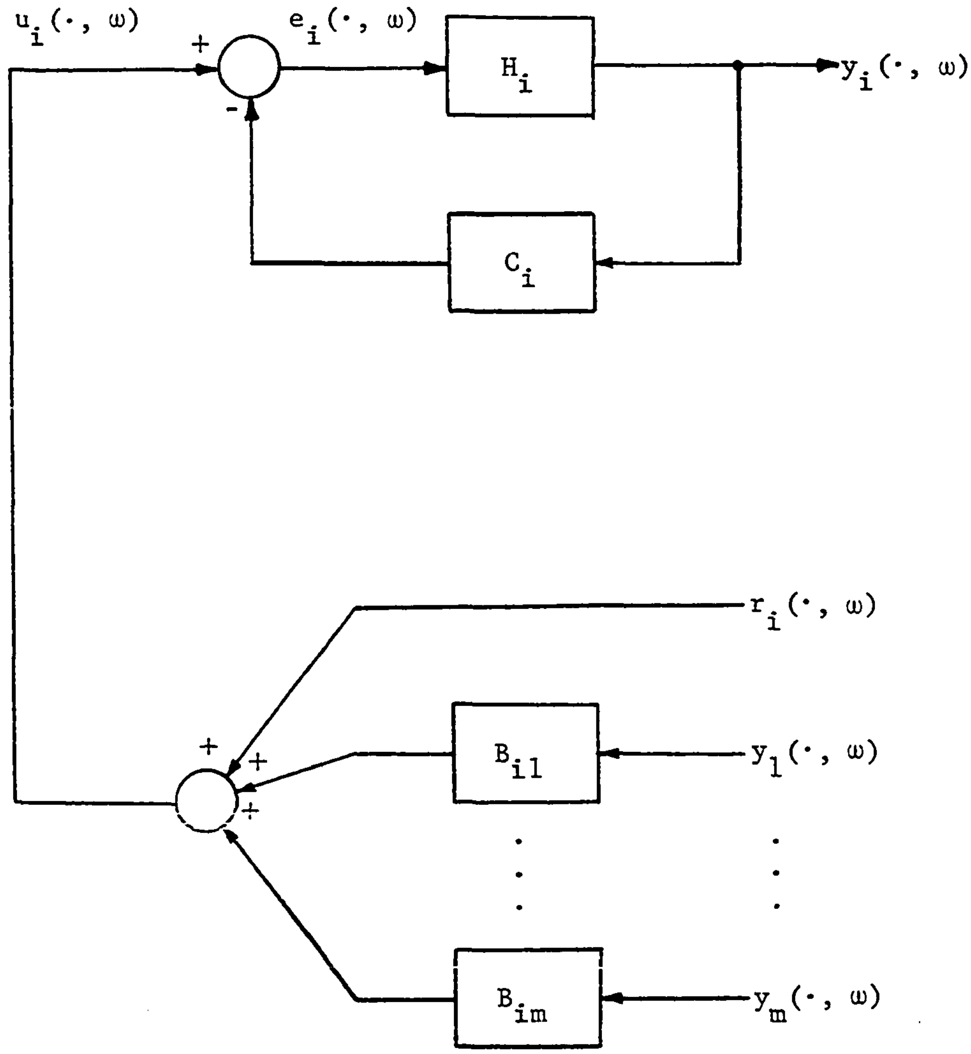

Fig. 3.1. Block diagram of System 3.1 or 3.3

where $\delta(t)$ is the Dirac delta function. Equation 3.5 may be rigorously written as

$$y_i(t, \omega) = \int_0^t w_i(t, s)e_i(s, \omega)d\beta_i(s), \tag{3.6}$$

where $\beta_i(s)$ is a generalized Wiener process with

$E\beta_i(t) = 0,\ t\epsilon R^+,\ i\epsilon M$, and

$E(d\beta_i(t))^2 = \sigma_i^2(t)dt,\ t\epsilon R^+,\ i\epsilon M.$

The integral in Eq. 3.6 is defined as an Ito integral (see Arnold [1] or Wong [42] for a discussion of the fundamental properties of the Ito integral). We also assume that the input process $\{r_i(t, \omega)\}$ from Eq. 3.1 or 3.2 is stochastically independent of $\{\beta_i(t, \omega)\}$, $t\epsilon R^+$.

Definition 3.2. Continuous-time Subsystem 3.2 is said to be of Type 1S if

(i) it is of Type 1;

(ii) $w_i(t, s)$ is time-invariant, that is,

$$w_i(t, s) = w_i(t - s) \triangleq w_i(\tau);$$

(iii) $w_i \epsilon L_1(R^+) \cap L_2(R^+)$;

(iv) $\beta_i(s)$ is a standard Wiener process with $\sigma_i^2(t) = \sigma_1^2,\ t\epsilon R^+$; and

(v) $B_{ii} = 0$, where 0 denotes the null operator on $S_{\infty e}$.

Definition 3.3. Discrete-time Subsystem 3.4 is said to be of Type 2 if

$$y_i(n, \omega) = \sum_{\ell=0}^{n-1} w_i(n, \ell) f_i(\ell, \omega) e_i(\ell, \omega), \qquad n \epsilon I^+,$$

where $w_i(n, \ell)$ is a discrete, real nonanticipatory convolution kernel, independent of ω, and where $f_i(n, \omega)$, $n \epsilon I^+$, is a sequence of independent second order random variables such that

$$Ef_i(n, \omega) = 0, \qquad n \epsilon I^+, \quad \text{and}$$

$$Ef_i(n, \omega) f_i(p, \omega) = \begin{cases} 0 & n \neq p \\ \sigma_i^2(n) & n = p, \quad n, p \epsilon I^+. \end{cases}$$

We assume that the input sequence $\{r_i(n, \omega)\}$, $n \epsilon I^+$, is stochastically independent of the sequence $\{f_i(n, \omega)\}$, $n \epsilon I^+$.

<u>Definition 3.4</u>. Discrete-time Subsystem 3.4 is said to be of <u>Type 2S</u> if

(i) it is of Type 2;

(ii) $w_i(n, \ell)$ is time invariant, that is,

$$w_i(n, \ell) = w_i(n - \ell) \triangleq w_i(k) \qquad n > \ell, \; n, \; \ell \epsilon I^+;$$

(iii) $\{f_i(n, \omega)\}$, $n \epsilon I^+$ is a weakly stationary stochastic process with $\sigma_i^2(n) = \sigma_i^2$, $n \epsilon I^+$, $\sigma_i^2 \epsilon R^+$; and

(iv) $B_{ii} = 0$, where 0 is the null operator on $S_{\infty e}$.

<u>Definition 3.5</u>. Continuous-time Subsystem 3.2 is said to be of <u>Type 3</u> if it is a Type 2S subsystem, except that

$$E\beta_i(t) = f_{oi}, \qquad t \epsilon R^+, \qquad f_{oi} \epsilon R, \text{ and}$$

$$E[d\beta_i(t) - f_{oi}dt]^2 = \sigma_i^2 dt, \quad t\epsilon R^+, \quad \sigma_i^2 \epsilon R^+.$$

Definition 3.6. Continuous-time Subsystem 3.2 is said to be of Type 4 if

(i) $y_i(t, \omega)$ is given by

$$y_i(t, \omega) = \int_0^t w_i(t - s)e_i(s, \omega)d\beta_i(s) + \int_0^t h_i(t - s)e_i(s, \omega)ds$$

where w_i, $h_i \epsilon L_1(R^+)$ and are time invariant, real nonanticipatory convolution operator kernels with Laplace transforms $\tilde{w}_i(s)$ and $\tilde{h}_i(s)$ respectively;

(ii) $\{\beta_i(s)\}$, $s\epsilon R^+$, is a standard Wiener process with

$$E\beta_i(t) = 0, \quad t\epsilon R^+, \text{ and}$$

$$E[d\beta_i(t)]^2 = \sigma_i^2 dt, \quad t\epsilon R^+, \quad \sigma_i^2 \epsilon R^+; \text{ and}$$

(iii) the integration in (i) with respect to the Wiener process is of the Ito type.

Definition 3.7. Continuous-time Subsystem 3.2 is said to be of Type 5 if it is of Type 4 with $w_i(t) = 0$ for $t\epsilon R^+$.

We will now define the type of stochastic stability investigated in the present chapter.

Definition 3.8. Continuous-time composite System 3.1 composed of subsystems of Type 1, 1S, 3, 4, or 5 is called second-order stochastic input-output stable if every input process vector, $r(t, \omega) \triangleq [r_1(t, \omega) \ldots, r_m(t, \omega)]^T$, $t\epsilon R^+$, with $r_i \epsilon S_\infty$, $i\epsilon M$, generates error and output process vectors $e(t, \omega) \triangleq [e_1(t, \omega), \ldots, e_m(t, \omega)]^T$ and $y(t, \omega) \triangleq [y_1(t, \omega), \ldots, y_m(t, \omega)]^T$, $t \in R^+$, such that $e_i, y_i \in S_\infty$ for all $i \in M$.

Definition 3.9. Discrete-time composite System 3.3, composed of subsystems of Type 2 or 2S, is called second-order stochastic input-output stable if every input process vector $r(n, \omega) \triangleq [r_1(n, \omega), \ldots, r_m(n, \omega)]^T$, $n \in I^+$, with $r_i \in s_\infty$, $i \in M$, generates error and output process vectors $e(n, \omega) \triangleq [e_1(n, \omega), \ldots, e_m(n, \omega)]^T$ and $y(n, \omega) \triangleq [y_1(n, \omega), \ldots, y_m(n, \omega)]^T$, $n \in I^+$, such that $e_i, y_i \in s_\infty$ for all $i \in M$.

Definition 3.10. Continuous-time composite System 3.1 or discrete-time composite System 3.3 is said to be second-order stochastic input-output unstable if it is not second order stochastic input-output stable. In the continuous-time case there exists at least one input process vector $r(t, \omega) = [r_1(t, \omega), \ldots, r_m(t, \omega)]^T$ that generates error process vector $e(t, \omega) = [e_1(t, \omega), \ldots, e_m(t, \omega)]^T$ and output process vector $y(t, \omega) = [y_1(t, \omega), \ldots, y_m(t, \omega)]^T$ such that for some $k \in M$ $e_k \in S_{\infty e} - S_\infty$ or $y_k \in S_{\infty e} - S_\infty$ (an analogous definition holds for the discrete-time case).

The above stability and instability definitions are an adaptation of similar definitions employed in [40].

3.3 Stability Results

The first two theorems presented in this section constitute the basic results of this chapter in the sense that the remaining theorems are essentially special cases of the first two. Theorems 3.1 and 3.2

are the composite stochastic system equivalent of the small gain theorems of deterministic single loop stability for continuous-time and discrete-time systems respectively.

Theorem 3.1. Continuous-time composite System 3.1 is second-order stochastic input-output stable if the following conditions hold:

(i) each isolated Subsystem 3.2 is of Type 1;

(ii) the Wiener processes $\beta_i(t)$ and $\beta_j(t)$ are mutually stochastically independent for all $i, j \in M$;

(iii) there exists $\alpha_i \in R^+$ for all $i \in M$ such that

$$\left[\int_0^t w_i^2(t, s)\sigma_i^2(s)ds\right]^{1/2} \leq \alpha_i, \quad t \in R^+; \text{ and}$$

(iv) all successive principal minors of the test matrix $A = [a_{ij}]$ are positive, where

$$a_{ij} = \begin{cases} 1 - \alpha_i(g_i - d_{ii}) & i = j \\ - \alpha_j d_{ij} & i \neq j \quad i, j \in M \end{cases}$$

(recall that g_i and d_{ij} represent bounds on the norms of the operators C_i and B_{ij}, respectively, on $S_{\infty e}$, as defined in Section 3.2).

Theorem 3.2. Discrete-time composite System 3.3 is second-order stochastic input-output stable if the following conditions hold:

(i) each isolated Subsystem 3.4 is of Type 2;

(ii) the stochastic processes $f_i(n, \omega)$ and $f_j(n, \omega)$, $n \in I^+$, are mutually stochastically independent for all $i, j \in M$, so that

$$Ef_i(n, \omega)f_j(p, \omega) = \begin{cases} 0 \quad n \neq p & i, j \in M \\ 0 \quad n = p & i \neq j \\ \sigma_i^2(n) & n = p, \quad i = j \end{cases}$$

for all n, $p \in I^+$ and i, $j \in M$;

(iii) there exists $\alpha_i \in R^+$ such that

$$\left[\sum_{\ell=0}^{n-1} w_i^2(n, \ell)\sigma_i^2(\ell)\right]^{1/2} \leq \alpha_i, \qquad n \in I^+$$

for all $i \in M$, and

(iv) all successive principal minors of the test matrix $A = [a_{ij}]$ are positive, where

$$a_{ij} = \begin{cases} 1 - \alpha_i(g_i - d_{ii}) & i = j \\ - \alpha_j d_{ij} & i \neq j, \quad i, j \in M, \end{cases}$$

where g_i and d_{ij} represent bounds on the norms of the operators C_i and B_{ij}, respectively, on $s_{\infty e}$, as defined in Section 3.2.

Remark 3.1. The test matrices in Theorems 3.1 and 3.2 are M-matrices (see Section 2.2). A necessary condition for these test matrices to have positive successive principal minors is that all elements of the principal diagonal be positive. If m = 1, then Theorems 3.1 and 3.2 reduce, essentially, to the single loop small gain results of Willems and Blankenship [40].

Remark 3.2. If, in Theorem 3.1, a particular isolated Subsystem 3.2, say the kth subsystem, is of Type 1S, then we may choose the parameter α_k as

$$\alpha_k^2 = \sigma_k^2 \int_0^\infty w_k^2(t)dt = \frac{\sigma_k^2}{2\pi} \int_{-\infty}^\infty |\tilde{w}_k(j\lambda)|^2 d\lambda,$$

where $\tilde{w}_k(j\lambda)$ is the Fourier transform of $w_k(t)$,

$$\tilde{w}_k(j\lambda) = \int_0^\infty w_k(t)e^{-j\lambda t}dt.$$

Similarly, if, in Theorem 3.2, some isolated Subsystem 3.4, say the k<u>th</u> subsystem, is of Type 2S, then we may choose the parameter α_k as

$$\alpha_k^2 = \sigma_k^2 \sum_{\ell=1}^\infty w_k^2(\ell) = \frac{\sigma_k^2}{2\pi} \int_{|z|=1} |\tilde{w}_k(z)|^2 dz,$$

where $\tilde{w}_k(z)$ is the z-transform of $w_k(n)$,

$$\tilde{w}_k(z) = \sum_{\ell=1}^\infty w_k(\ell)z^{-\ell}.$$

The next theorem applies to continuous-time systems and allows us to model the multiplicative noise in a subsystem as a constant, or bias term, plus white noise. This may be used as a basis for many of the models mentioned in Section 3.1.

<u>Theorem 3.3</u>. Continuous-time composite System 3.1 is second-order stochastic input-output stable if the following conditions hold:

(i) each isolated Subsystem 3.2 is of Type 1, 1S, or 3;

(ii) if the k<u>th</u> Subsystem 3.2 is of Type 3, then $w_i \epsilon L_1(R^+) \cap L_2(R^+)$ and w_i has Property L (see Section 2.1);

(iii) the Wiener processes $\beta_i(t)$ and $\beta_j(t)$ are mutually stochastically independent for all i, jϵM;

(iv) if the kth Subsystem 3.2 is of Type 1 or 1S, then there exists $\alpha_k \epsilon R^+$ such that

$$\left[\int_0^t w_k^2(t, s)\sigma_k^2(s)ds\right]^{1/2} \leq \alpha_k, \qquad t\epsilon R^+,$$

and if the kth Subsystem 3.2 is of Type 3, then there exists $\alpha_k \epsilon R^+$ such that

$$\left[(\sigma_k^2/2\pi)\int_{-\infty}^{\infty} |\tilde{w}_k(j\lambda)/(1 + f_{ok}\tilde{w}_k(j\lambda))|^2 d\lambda\right]^{1/2} \leq \alpha_k$$

where

$$\tilde{w}_k(j\lambda) = \int_0^{\infty} w_k(t)e^{-j\lambda t}dt$$

is the Fourier transform of $w_k(t)$;

(v) there exists $\gamma_k \epsilon R^+$ such that $\gamma_k = 1$ if the kth Subsystem 3.2 is of Type 1 or 1S and

$$1 + \int_0^{\infty} |\bar{r}_k(t)|dt \leq \gamma_k$$

where $\bar{r}_k$ is the resolvent of $w_k(t)$, if the kth Subsystem 3.2 is of Type 3; and

(vi) all successive principal minors of the test matrix $A = [a_{ij}]$ are positive, where

$$a_{ij} = \begin{cases} 1 - \alpha_i g_i - \alpha_i d_{ii} & i = j \\ \gamma_i \alpha_j d_{ij} & i \neq j, \quad i, j\epsilon M. \end{cases}$$

Remark 3.3. If the kth Subsystem 3.2 is of Type 3 then α_k may be determined graphically by Corollary 3 of [40]. In particular, if for the kth Subsystem 3.2 of Type 3, there exists an $a_k \epsilon R$ such that $\sigma_k^2 w_k(0)/(f_{ok} + a_k) < 1$, and if one of the following cases pertain:

(i) if $f_{ok}/a_k > 0$ and the Nyquist plot of $\tilde{w}_k(j\lambda)$ lies inside the circle which passes through the origin and the point $(1/a_k, 0)$ and which is symmetric with respect to the real axis; or

(ii) if $-1 < f_{ok}/a_k < 0$ and the Nyquist plot of $\tilde{w}_k(j\lambda)$ lies inside the circle which passes through the origin and the point $(1/a_k, 0)$ and which is symmetric with respect to the real axis; or

(iii) if $f_{ok}/a_k < -1$ and the Nyquist plot of $\tilde{w}_i(j\lambda)$ does not encircle or intersect the circle which passes through the origin and the point $(1/a_k, 0)$ and is symmetric with respect to the real axis;

then we may choose

$$\alpha_k = \frac{\sigma_k^2 w_k(0)}{f_{ok} + a_k} .$$

Remark 3.4. For a Type 3 Subsystem 3.2, the parameter γ_k of Theorem 3.3 may be determined from $\tilde{w}_k(s)$, where

$$\tilde{w}_k(s) = \int_0^\infty w_k(t) e^{-st} dt,$$

if $f_{ok} w_k(t) \geq 0$, $t \epsilon R^+$, by noting in this case that the resolvent associated with $w_k(t)$, $\bar{r}_k(t)$, is nonnegative for $t \epsilon R^+$, and

$$\int_0^\infty |\bar{r}_k(t)| dt = \left| \lim_{s \to 0} \frac{f_{ok} \tilde{w}_k(s)}{1 + f_{ok} \tilde{w}_k(s)} \right| .$$

This is also true for $w_k(t) \leq 0$, $t \epsilon R^+$.

Theorem 3.4. Continuous-time composite System 3.4 is second-order stochastic input-output stable if the following conditions hold:

(i) each isolated Subsystem 3.2 is of Type 1, 1S, 3, 4, or 5;

(ii) the Wiener processes $\beta_i(t)$ and $\beta_j(t)$ are mutually stochastically independent for all i, $j\epsilon M$;

(iii) hypotheses (iv) and (v) of Theorem 3.3 are true;

(iv) if the kth Subsystem 3.2 is of Type 4, then $h_k(t)$ has Property L and $w_k \epsilon L_1(R^+) \cap L_2(R^+)$;

(v) if the kth Subsystem 3.2 is of Type 4, then there exist α_k, $\gamma_k \epsilon R^+$ such that

$$\left[\frac{\sigma_k^2}{2\pi}\int_{-\infty}^{\infty}\left|\frac{\tilde{w}_k(j\lambda)}{1+\tilde{h}_k(j\lambda)}\right|^2 d\lambda\right]^{1/2} \leq \alpha_k, \text{ and}$$

$$1 + \int_0^{\infty} |\bar{r}_k(t)|dt \leq \gamma_k$$

where $\bar{r}_k(t)$ is the resolvent of $h_k(t)$;

(vi) if the kth Subsystem 3.2 is of Type 5 then there exists $\alpha_k \epsilon R^+$ such that

$$\int_0^{\infty} |h_k(t)|dt \leq \alpha_k$$

and $\gamma_k = 1$; and

(vii) all successive principal minors of the test matrix $A = [a_{ij}]$ are positive, where

$$a_{ij} = \begin{cases} 1 - \alpha_i g_i - \alpha_i d_{ii} & i = j \\ - \gamma_i \alpha_j d_{ij} & i \neq j \quad i, j\epsilon M. \end{cases}$$

<u>Remark 3.5</u>. The comments in Remarks 3.3 and 3.4 hold for Theorem 3.4 as well. Also, if the <u>kth</u> Subsystem 3.2 is of Type 5 and $h_k(t) \geq 0$ (≤ 0), $t \epsilon R^+$, then the parameter α_k may be computed as

$$\alpha_k = |\lim_{s \to 0} \tilde{h}_k(s)|.$$

3.4 Instability Results

In the following, instability results are established for continuous-time systems composed of Type 1S subsystems and for discrete-time system composed of Type 2S subsystems. Furthermore, we restrict the interconnecting and feedback operators B_{ij} and C_i respectively to be of the form

$$\left.\begin{aligned} C_i y_i(t, \omega) &= \overline{c_i} y_i(t, \omega) \\ B_{ij} y_j(t, \omega) &= \overline{b_{ij}} y_j(t, \omega) \end{aligned}\right\} \qquad (3.7)$$

where $\overline{c_i}$, $\overline{b_{ij}} \epsilon R$ and y_i, $y_j \epsilon S_{\infty e}$ for continuous-time systems. C_i and B_{ij} are restricted similarly for discrete-time systems. These operators, therefore, represent constant multipliers.

In accordance with [40], if an isolated subsystem described by Eq. 3.2 is of Type 1S and if $|\overline{c_i}|\alpha_i > 1$, then the subsystem is second order stochastic input-output unstable in the sense that there exists at least one input process $r_i \epsilon S_{\infty e} - S_\infty$ such that the error process $e_i \epsilon S_{\infty e} - S_\infty$. Similarly if an isolated subsystem described by Eq. 3.4 is of Type 2S and if $|\overline{c_i}|\alpha_i \geq 1$, then there exists at least one input sequence $r_i \epsilon s_\infty$ such that the error process $e_i \epsilon s_{\infty e} - s_\infty$. In the subsequent results we show that if, under appropriate conditions, one subsystem is unstable in the above sense, then the entire composite system will also be second-order stochastic input-output unstable.

Theorem 3.5. Continuous-time composite System 3.1 is second-order stochastic input-output unstable if the following conditions hold:

(i) each isolated Subsystem 3.2 is of Type 1S;

(ii) operators C_i and B_{ij} are characterized by Eq. 3.7;

(iii) the Wiener processes $\beta_i(t)$ and $\beta_j(t)$ are mutually stochastically independent for all i, $j \epsilon M$;

(iv) $r_i(t, \omega)$ and $\beta_j(s, \omega)$ are mutually stochastically independent for all i, $j \epsilon M$, and s, $t \epsilon R^+$;

(v) for the kth subsystem, for some $k \epsilon M$, the inequality

$$|\overline{c_k}|^2 \sigma_k^2 \int_0^\infty w_k^2(t)dt = [(|\overline{c_k}|^2 \sigma_k^2)/2\pi] \int_{-\infty}^\infty |\tilde{w}_k(j\lambda)|^2 d\lambda \geq 1$$

holds; and

(vi) $w_k(t)$ is continuous, $t \epsilon R^+$, and k is as defined in (v) above.

Theorem 3.6. Discrete-time composite System 3.3 is second-order stochastic input-output unstable if the following conditions hold:

(i) each isolated Subsystem 3.4 is of Type 2S;

(ii) operators C_i and B_{ij}, i, $j \epsilon M$, are of the type described above;

(iii) the stochastic processes $f_i(n, \omega)$ and $f_j(n, \omega)$ are mutually stochastically independent for all i, $j \epsilon M$;

(iv) $r_i(n, \omega)$ and $f_j(p, \omega)$ are mutually stochastically independent for all $i, j \epsilon M$ and $n, p \epsilon I^+$; and

(v) for the k<u>th</u> subsystem, for some $k \epsilon M$, the inequality

$$(\overline{c_k}\sigma_k)^2 \sum_{\ell=1}^{\infty} w_k^2(\ell) = [(\overline{c_k}\sigma_k)^2/2\pi] \int_{|z|=1} |\tilde{w}_k(z)|^2 dz \geq 1$$

holds.

4. SECOND ORDER STOCHASTIC ABSOLUTE STABILITY

4.1 Introduction

As in the previous chapter, systems with gain terms which may be modeled as stochastic processes are investigated in the present chapter. Unlike Chapter 3, however, the systems in this chapter are endowed with a nonlinear element in the forward path of each loop. This nonlinearity sufficiently complicates the system so that only relatively basic results have been obtained. The results presented are for a single loop system (as opposed to an interconnected system) and involve a definition of stochastic stability somewhat different from the one employed in Chapter 3. For stability we require that the second moment of the error and output processes exist for each $t \in R^+$ and that the second moments of these processes tend to zero as t becomes large. In this respect we establish frequency-domain results reminiscent of the familiar circle criterion of deterministic stability theorems (for a summary of the original and fundamental input-output frequency-domain stability results due to Sandberg and Zames, refer to Desoer and Vidyasagar [5] and Willems [39]). The results of the present chapter are for continuous-time systems exclusively. Background material is presented in the next section. In the following section the main results are presented. An example demonstrating the utility of the results of this chapter is included in Chapter 6 (Example 6.4). Proofs of Theorem 4.1 and Corollary 4.1 presented here appear in Appendix B.

4.2 Preliminaries

In this chapter (and in Appendix B) we use the symbols $\|\cdot\|$ and $\|\cdot\|_T$ in an $L_2(R^+)$ sense, that is,

$$\|x(t)\| = \left(\int_0^\infty x^2(t)dt\right)^{1/2}, \quad x\epsilon L_2(R^+), \text{ and}$$

$$\|x(t)\|_T = \left(\int_0^T x^2(t)dt\right)^{1/2}, \quad x\epsilon L_{2e}(R^+),\ T\epsilon R^+.$$

A similar convention is used for operator norms on these spaces. We consider continuous-time systems that may be modeled (informally) by the following stochastic integral equation:

$$\left.\begin{aligned} e(t, \omega) &= u(t, \omega) - y(t, \omega) \\ y(t, \omega) &= \int_0^t g(t - \tau)\psi(e(\tau, \omega), \tau)f(\tau, \omega)d\tau \end{aligned}\right\}, \tag{4.1}$$

where it is assumed that $(Ee^2(t, \omega))$, $(Ey^2(t, \omega))$, and $(Eu^2(t, \omega))$ belong to $L_{2e}(R^+)$; the convolution kernel, g, is real, nonanticipatory, and belongs to $L_2(R^+)$; ψ is a memoryless nonlinearity satisfying

$$0 < a \leq \frac{\psi(x, t)}{x} \leq b < \infty, \quad x\epsilon R,\ t\epsilon R^+;$$

and $f(t, \omega)$ denotes a white noise process with

$$Ef(t, \omega) = 0, \qquad t\epsilon R^+, \text{ and}$$

$$E\left\{f(t, \omega)f(t + \tau, \omega)\right\} = \sigma^2\delta(\tau), \quad t, \tau, \epsilon R^+,$$

where $\delta(\tau)$ denotes the Dirac delta function. Let $e(t, \omega)$ denote the error process, $y(t, \omega)$ the output process, and $u(t, \omega)$ the input process. Figure 4.1 depicts a system modeled by Eq. 4.1. This equation may be rigorously written as

$$e(t, \omega) = u(t, \omega) - \int_0^t g(t - \tau)\psi(e(\tau, \omega)\tau)d\beta(\tau), \qquad (4.2)$$

where $\beta(t)$ is a Wiener process with

$$E\beta(t) = 0, \qquad t\varepsilon R^+, \text{ and}$$

$$E[d\beta(t)]^2 = \sigma^2 dt.$$

The integral in Eq. 4.2 is defined as an Ito integral. We also assume that the system input, $u(t, \omega)$, and the Wiener process $\beta(t)$ are stochastically independent for $t\varepsilon R^+$.

We now give a precise definition of the type of stochastic stability considered in the present chapter.

Definition 4.1. The continuous-time System 4.1 is said to be second-order stochastically absolutely input-output stable if every input process $\{u(t, \omega)\}$, $t\varepsilon R^+$, whose second order statistics satisfy

$$\{(Eu^2(t, \omega))\} \ \varepsilon \ L_2(R^+), \text{ and}$$

$$Eu^2(t, \omega) \rightarrow 0 \text{ as } t \rightarrow \infty,$$

generates error and output processes, $\{e(t, \omega)\}$, and $\{y(t, \omega)\}$ respectively, whose second order statistics similarly satisfy

$$\{(Ee^2(t, \omega))\} \ \varepsilon \ L_2(R^+),$$

$$Ee^2(t, \omega) \rightarrow 0 \text{ as } t \rightarrow \infty,$$

$$\left\{(Ey^2(t,\omega))\right\} \in L_2(R^+), \text{ and}$$

$$Ey^2(t,\omega) \to 0 \text{ as } t \to \infty.$$

4.3 Main Results

The proofs of the following results are presented in Appendix B.

Theorem 4.1. The continuous-time System 4.1 is second-order stochastically absolutely input-output stable if the following conditions are met:

(i) $g^2 \in L_1(R^+) \cap L_2(R^+)$;

(ii) $1 - (\sigma^2/2)(a^2 + b^2)G_2(s) \neq 0$ for $Re(s) \geq 0$, where $G_2(s)$ is the Laplace transform of $g^2(t)$; and

(iii) $$\sup_{\lambda \in R} \left| \frac{G_2(j\lambda)}{1 - (\sigma^2/2)(a^2 + b^2)G_2(j\lambda)} \right| \frac{\sigma^2}{2} (b^2 - a^2) < 1.$$

Theorem 4.1 may be recast in terms of the Nyquist plot of $G_2(s)$ as in the following result.

Corollary 4.1. The continuous-time System 4.1 is second-order stochastically absolutely input-output stable if the following conditions are met:

(i) $g^2 \in L_1(R^+) \cap L_2(R^+)$; and

(ii) the locus of $G_2(j\lambda)$, $\lambda \in R$, where $G_2(s)$ is the Laplace transform of $g^2(t)$, does not encircle or intersect the circle in the complex plane with center $((1/2\sigma^2)(\frac{1}{a^2} + \frac{1}{b^2}), 0)$ and radius $(1/2\sigma^2)(\frac{1}{a^2} - \frac{1}{b^2})$. Figure 4.2 depicts such a circle in the complex plane.

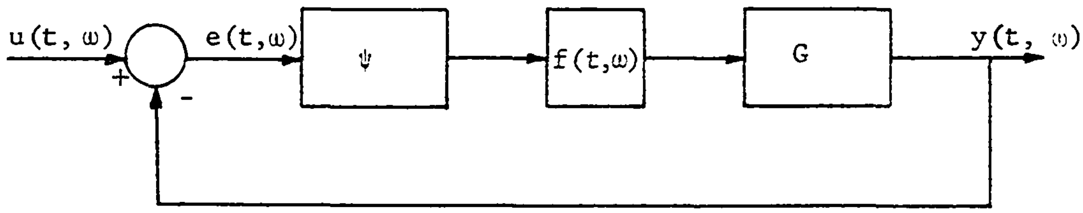

Fig. 4.1. Block diagram of System 4.1

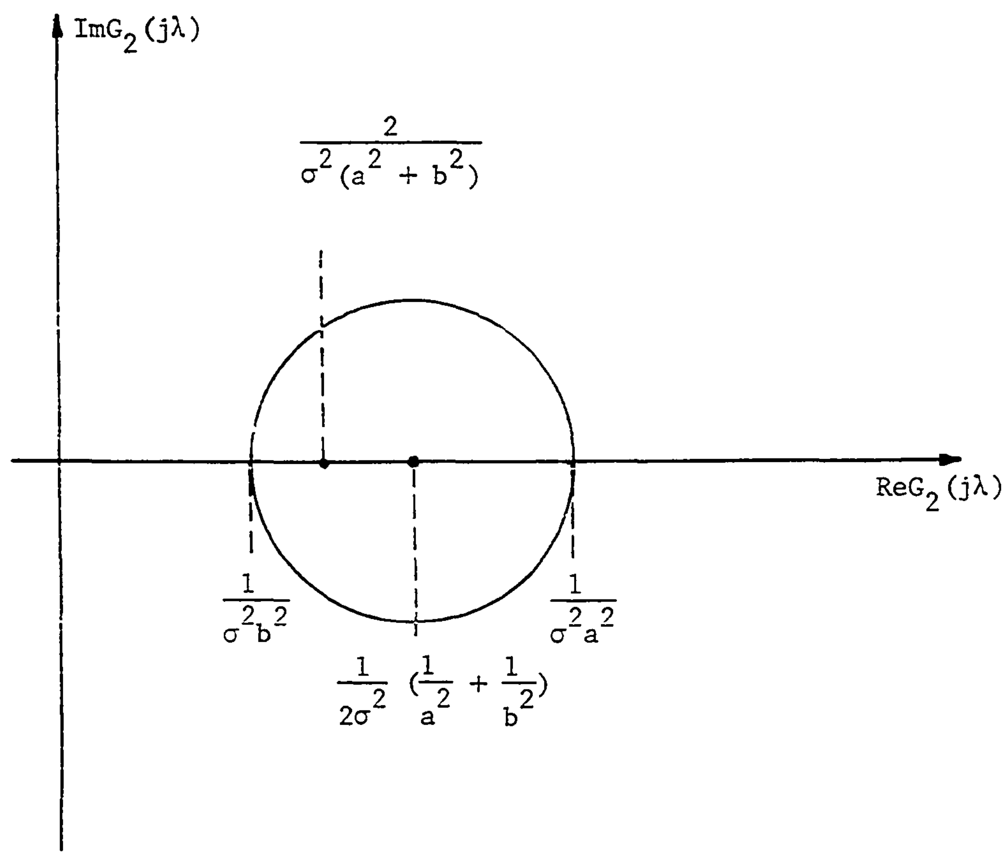

Fig. 4.2. Location of the circle for the corollary

Remark 4.1. If G(s), the Laplace transform of g(t), is a rational function of s, G(s) = A(s)/B(s), and has only q first order poles, then $G_2(s)$, the Laplace transform of $g^2(t)$, may be computed directly from G(s). Condition (ii) of Corollary 4.1 becomes (see McCollum and Brown [20])

(ii') the locus of

$$\sum_{k=1}^{q} \frac{A(s_k)A(j\lambda - s_k)}{B'(s_k)B(j\lambda - s_k)}, \quad \lambda \epsilon R,$$

does not encircle or intersect a circle in the complex plane with center $((1/2\sigma^2)(\frac{1}{a^2} - \frac{1}{b^2}), 0)$ and radius $(1/2\sigma^2)(\frac{1}{a^2} - \frac{1}{b^2})$, where s_k is the kth pole location of G(s) and

$$B'(s_k) = \frac{d}{ds} B(s)\Big|_{s=s_k} .$$

Remark 4.2. If G(s) is a rational function of s having n poles, where the pole at s_k is of order m_k, then $G_2(s)$ may be computed from (see McCollum and Brown [20])

$$G_2(s) = \sum_{k=1}^{n} \sum_{s=1}^{m_k} \frac{(-1)^{m_k-1} K_{kj}}{(m_k - j)!} \left[\frac{d^{m_k-j}}{d\lambda^{m_k-j}} G(\lambda)\right]_{\lambda=s-s_k}$$

where

$$K_{kj} = \frac{1}{(j-1)!} \left[\frac{d^{j-1}}{ds^{j-1}} (s - s_k)^{m_k} G(s)\right]_{s=s_k} .$$

Remark 4.3. By an application of the Chebyshev inequality it follows that $e(t, \omega)$ and $y(t, \omega)$ converge in probability to zero, that is, given an $\epsilon > 0$, there exists a $T^* \epsilon R^+$ such that for $t \geq T^*$, $P[e(t, \omega) > \epsilon] < \epsilon$, and similarly for $y(t, \omega)$. In the linear case, it was shown in [40] that if G(s) is a rational function of s and finite dimensional and if $\sigma\|g\| < 1$, then System 4.1 is Lyapunov stable with probability one. In the present nonlinear situation this implication does not necessarily follow. No general relationship implying Lyapunov stability with probability one from the second moment input-output stabilities of System 4.1 is known to exist at this time.

5. STOCHASTIC ABSOLUTE STABILITY

5.1 Introduction

In this chapter we establish new results for the stability of large scale systems described by nonlinear Volterra integral equations with random driving functions and random coefficients. Systems with random inputs and coefficients have long been the subject of study as can be seen by reading the survey paper by Kozin [11]. Such systems are still of interest as evidenced by the recent works of Morozan [28], Tsokos [37], and Tsokos and Padgett [38].

As in Chapter 3, we are interested in determining the stability of high dimensional systems from properties of lower order subsystems and the interconnecting structure. Models of large scale electrical networks, large scale economic and political systems, models of ecological and biological systems, and models of social systems all are candidates for the analysis presented in this chapter.

Unlike the previous chapters we are not concerned here with the behavior of the second moments of a system. We are concerned with the behavior of the sample path with probability one. Specifically the error and output processes are required to tend to zero almost surely as time becomes large. As a consequence of the approach taken, we also establish that the sample paths of the error and output processes are square integrable over R^+ with probability one. Also, unlike in the previous chapters, we are not concerned here with input-output stability in the strict sense of the term. That is, we do not establish an input space and an output space and claim that bounded inputs (bounded with respect to the input space) produce bounded outputs (with respect to the output space). In this respect the material herein is closer to the Liapunov-type stability with probability one results of Kushner [12], Michel and Rasmussen [24, 31] and others. However, it should be noted that the following results pertain to driven systems, while the standard Liapunov results require either an undriven system (zero input) or exact prior knowledge of the driving function.

For the special case where the underlying probability space becomes trivial (that is, when the stochastic system reduces essentially to a deterministic system) the resulting stability theorems for large scale deterministic systems have not previously been established.

The composite system results presented here are based primarily on the single-loop results developed by Sandberg [32, 35] for deterministic systems, by Tsokos [37] and Tsokos and Padgett [38] for stochastic systems and on the deterministic composite system results of Lasley and Michel [14].

Mathematical notations and preliminaries are introduced in the next section. In the following section the main results are presented, while in the fourth section the main results are applied to systems described by differential equations with random coefficients. A control system example using the techniques of Section 5.3 is provided in Chapter 6 (Example 6.5). All results are proved in Appendix C.

5.2 Mathematical Background

In this chapter (and in Appendix C) we use the symbols $\|\cdot\|$ and $\|\cdot\|_T$ in an $L_{2(N_i)}(R^+)$ sense, that is

$$\|x(t)\| = \left[\int_0^\infty |x(t)|^2 dt\right]^{1/2}, \quad x \epsilon L_{2(N_i)}(R^+), \text{ and}$$

$$\|x(t)\|_T = \left[\int_0^T |x(t)|^2 dt\right]^{1/2}, \quad x \epsilon L_{2e(N_i)}(R^+), \; T \epsilon R^+.$$

A similar convention is used for operator norms on these spaces. We consider composite systems described by the following stochastic integral operator equations:

$$\left.\begin{aligned} e_i(t, \omega) &= u_i(t, \omega) - y_i(t, \omega). \\ y_i(t, \omega) &= \int_0^t k_i(t - \tau, \omega)\psi_i(e_i(\tau, \omega), \tau, \omega)d\tau \\ u_i(t, \omega) &= r_i(t, \omega) + \sum_{j=1}^m B_{ij}e_j(t, \omega) + \sum_{j=1}^m D_{ij}y_j(t, \omega) \end{aligned}\right\} \quad (5.1)$$

$i, j \epsilon M = \{1, 2, \ldots m\}$. For each $i \epsilon M$, we assume that r_i, e_i, y_i, and u_i belong to $E_{2(N_i)}$; $\psi_i \epsilon \eta_{(N_i)}$; $k_i \epsilon K_{1(N_i \times N_i)}$. For each $i, j \epsilon M$ B_{ij} and D_{ij} are operators on $E_{2(N_i)}$ with values in $E_{2(N_j)}$. These operators are assumed to be one of two types: either a Type A operator with

$$B_{ij}e_j(t, \omega) = b_{Aij}(t, \omega) \cdot e_j(t, \omega), \text{ or}$$

$$D_{ij}y_j(t, \omega) = d_{Aij}(t, \omega) \cdot y_j(t, \omega),$$

where $b_{Aij}(t, \omega)$ and $d_{Aij}(t, \omega)$ are $N_i \times N_j$-dimensional matrix-valued random processes with elements in $L_{2(1)}(R^+, L_\infty(\Omega))$; or a Type B operator with

$$B_{ij}e_j(t, \omega) = \int_0^t b_{Bij}(t - \tau, \omega)\xi_{ij}(e_j(\tau, \omega), \tau, \omega)d\tau, \text{ or}$$

$$D_{ij}y_j(t, \omega) = \int_0^t d_{Bij}(t - \tau, \omega)\xi_{ij}(y_j(\tau, \omega), \tau, \omega)d\tau,$$

where b_{Bij} and d_{Bij} belong to $K_{1(N_i \times N_j)}(R^+, L_\infty(\Omega)) \cap K_{2(N_i \times N_j)}(R^+, L_\infty(\Omega))$ and $\xi_{ij} \epsilon \eta_{(N_1)}$. We define the following $\sum_{j=1}^m N_j \times \sum_{j=1}^m N_j$ – dimensional matrices of operators:

$$B_A = [B_{Aij}]$$

where

$$B_{Aij} = \begin{cases} B_{ij} & \text{if } B_{ij} \text{ is of Type A} \\ 0 & \text{if } B_{ij} \text{ is of Type B}, \quad i, j \epsilon M \end{cases}$$

and

$$B_B = [B_{Bij}]$$

where

$$B_{Bij} = \begin{cases} B_{ij} & \text{if } B_{ij} \text{ is of Type B} \\ 0 & \text{if } B_{ij} \text{ is of Type A}, \quad i, j \epsilon M. \end{cases}$$

We also define the operators K_i and Q_i on $E_{2(N_i)}$, $i \epsilon M$, by

$$K_i x(t, \omega) = \int_0^t k_i(t - \tau, \omega) x(\tau, \omega) d\tau, \quad t \epsilon R^+, \; x \epsilon E_{2(N_i)}$$

and

$$Q_i x(t, \omega) = \psi_i(x(t, \omega), \tau, \omega), \quad t \epsilon R^+, \; x \epsilon E_{2(N_i)}.$$

Furthermore we define the symbol $\tilde{K}_i(s, \omega)$ by

$$\tilde{K}_i(s, \omega) = \int_0^\infty k_i(t, \omega) e^{-st} dt.$$

Recall that an operator H on $E_{2(N)}$ is causal if, for any arbitrary $T \epsilon R^+$,

$$\pi_T H x(t) = \pi_T H \pi_T x(t), \quad t \epsilon R^+, \; x \epsilon E_{2(N)}$$

where π_T is the truncation operator $(\pi_T x(t) = x_T(t))$. It is assumed in this chapter that K_i, Q_i, B_{ij}, and D_{ij} are causal operators for all i and j.

System 4.1 may be viewed as the interconnection of m free or isolated subsystems, each of dimension N_i and each described by an equation of the form

$$e_i(t, \omega) = r_i(t, \omega) - \int_0^t k_i(t - \tau, \omega) \psi_i(e_i(\tau, \omega), \tau, \omega) d\tau, \quad i \epsilon M. \qquad (5.2)$$

We now define the type of stochastic stability we will be concerned with in this chapter.

Definition 5.1. Continuous-time System 5.1 is said to be stochastically absolutely stable if

$$P\left\{\omega: \lim_{t\to\infty} e_i(t, \omega) = 0\right\} = 1, \qquad i\epsilon M$$

and

$$P\left\{\omega: \lim_{t\to\infty} y_i(t, \omega) = 0\right\} = 1, \qquad i\epsilon M.$$

5.3 Main Results

The following theorems are proved in Appendix C.

Theorem 5.1. Continuous-time System 5.1 is stochastically absolutely stable if the following conditions hold:

(i) $r_i \epsilon L_{2(N_i)}(R^+, L_\infty(\Omega))$ and $|r_i(t, \omega)| \to 0$ as $t \to \infty$ a.e.[P], $i\epsilon M$;

(ii) $\det[I + \frac{1}{2}(a_i + b_i)\tilde{K}_i(s, \omega)] \neq 0$ for $\mathrm{Re}(s) \geq 0$, a.e.[P], $i\epsilon M$;

(iii) the test matrix $A = [a_{ij}]$ has positive successive principal minors, where

$$a_{ik} = \begin{cases} 1 - \alpha_i - \gamma_{ii} - \xi_{ii}\mu_i & i = k \\ -\gamma_{ik} - \xi_{ik}\mu_k & i \neq k \quad i,k\epsilon M \end{cases}$$

with

$$\alpha_i \geq \frac{1}{2}(b_i - a_i) \sup_{\lambda\epsilon R^+} [(I + \frac{1}{2}(a_i + b_i)\tilde{K}_i(j\lambda, \omega))^{-1}\tilde{K}_i(j\lambda, \omega)]$$

a.e.[P], $i\epsilon M$,

$$\gamma_{ik} \geq \| (I + \frac{1}{2} (a_i + b_i)K_i)^{-1} B_{ik} \| \qquad \text{a.e.}[P], \quad i, k \epsilon M,$$

$$\xi_{ik} \geq \| (I + \frac{1}{2} (a_i + b_i)K_i)^{-1} D_{ik} \| \qquad \text{a.e.}[P], \quad i, k \epsilon M,$$

$\mu = \max (|a_i|, |b_i|)$; and

(iv) the operator $(I - B_A)$ has a bounded inverse for $t \geq T^*$ a.e.[P] for some $T^* \epsilon R^+$.

Remark 5.1. If B_{ik} or D_{ik} are of the form

$$B_{ik} e_k(t, \omega) = \int_0^t b_{ik}(t - \tau, \omega) e_k(\tau, \omega) d\tau, \qquad t \epsilon R^+,$$

or

$$D_{ik} y_k(t, \omega) = \int_0^t d_{ik}(t - \tau, \omega) y_k(\tau, \omega) d\tau, \qquad t \epsilon R^+,$$

with $b_{ik}, d_{ik} \epsilon K_{1(N_k X N_i)}(R^+, L_\infty(\Omega))$, then the A-matrix elements γ_{ik} or ξ_{ik} may be found from

$$\gamma_{ik} \geq \sup_{\lambda \epsilon R^+} \Lambda \left\{ (I + \frac{1}{2} (a_i + b_i) \tilde{K}(j\lambda, \omega))^{-1} \tilde{B}_{ik}(j\lambda, \omega) \right\} \quad \text{a.e.}[P],$$

or

$$\xi_{ik} \geq \sup_{\lambda \epsilon R^+} \Lambda \left\{ (I + \frac{1}{2} (a_i + b_i) \tilde{K}_i(j\lambda, \omega))^{-1} \tilde{D}_{ik}(j\lambda, \omega) \right\} \text{a.e.}[P],$$

where $\tilde{K}_i(s, \omega)$, $\tilde{B}_{ik}(s, \omega)$, and $D_{ik}(s, \omega)$ represent the Laplace transforms of $k_i(t, \omega)$, $b_{ik}(t, \omega)$, and $d_{ik}(t, \omega)$ respectively. For the $N_i = N_k = 1$ case, the above A-matrix elements may be determined graphically. It can be seen that under these conditions γ_{ik} is the smallest number, b, such that the locus of $\frac{1}{b} \tilde{B}_{ik}(j\lambda, \omega)$, $\lambda \epsilon R$, is inside the locus of $(1 + \frac{1}{2}(a_i + b_i)\tilde{K}_i(j\lambda, \omega))$, $\lambda \in R$, for almost every ω. For further frequency-domain interpretations see Remark 5.4.

Remark 5.2. The deterministic version of this theorem (which has not previously appeared) may be obtained by taking $\Omega = \{1\}$ and $P\{1\} = 1$.

Remark 5.3. For $m = N_i = 1$, $B_{11} = D_{11} = 0$, and for the deterministic case (see Remark 5.2), Theorem 5.1 reduces to a version of the familiar circle theorem introduced by Sandberg [33] and Zames [44, 45].

Remark 5.4. For $N_i = 1$ the A-matrix terms α_i may be determined from the Nyquist locus of $\tilde{K}_i(j\lambda, \omega)$. Note that we desire to find an α_i such that for almost every ω

$$\frac{1}{2}(b_i - a_i)|\tilde{K}_i(j\lambda, \omega)| \leq \alpha_i |1 + \frac{1}{2}(a_i + b_i)\tilde{K}_i(j\lambda, \omega)|$$

or

$$\left(\frac{b_i - a_i}{2\alpha_i}\right)^2 |\tilde{K}_i(j\lambda, \omega)|^2 \leq |1 + \frac{1}{2}(a_i + b_i)\tilde{K}_i(j\lambda, \omega)|^2,$$

that is,

$$\left(\frac{b_i - a_i}{2\alpha_i}\right)^2 \tilde{K}_i(j\lambda, \omega)\tilde{K}_i^*(j\lambda, \omega) \leq (1 + \frac{1}{2}(a_i + b_i)\tilde{K}_i(j\lambda, \omega)) \cdot (1 + \frac{1}{2}(a_i + b_i)\tilde{K}_i^*(j\lambda, \omega)).$$

We may write, for almost every $\omega \in \Omega$

$$0 \leq 1 + \frac{1}{2}(a_i + b_i)\tilde{K}_i + \frac{1}{2}(a_i + b_i)\tilde{K}_i^* + \left[\left(\frac{a_i + b_i}{2}\right)^2 - \left(\frac{b_i - a_i}{2\alpha_i}\right)^2\right]\tilde{K}_i\tilde{K}_i^*$$

where the arguments of $\widetilde{K}_i (\widetilde{K}_i^*)$ are assumed to be $(j\lambda, \omega)$. Defining

$$\left.\begin{aligned} \Delta_i &= \frac{b_i - a_i}{2\alpha_i} \\ \sigma_i &= \frac{a_i + b_i}{2} \\ \text{and} \qquad \rho_i &= \sigma_i^2 - \Delta_i^2 \end{aligned}\right\}, \tag{5.3}$$

we have

$$0 \leq 1 + \sigma_i \widetilde{K}_i + \sigma_i \widetilde{K}_i^* + \rho_i \widetilde{K}_i \widetilde{K}_i^* \ . \tag{5.4}$$

We now consider three cases.

(a) If $\rho_i > 0$, Eq. 5.4 may be written, for almost every $\omega \epsilon \Omega$, as

$$0 \leq \frac{1}{\rho_i} + (\frac{\sigma_i}{\rho_i})\widetilde{K}_i + (\frac{\sigma_i}{\rho_i})\widetilde{K}_i^* + \widetilde{K}_i \widetilde{K}_i^*$$

or equivalently as

$$|\widetilde{K}_i + \frac{\sigma_i}{\rho_i}| \geq [(\frac{\sigma_i}{\rho_i})^2 - (\frac{1}{\rho_i})]^{1/2} = |\frac{\Delta_i}{\rho_i}| \ . \tag{5.5}$$

Equation 5.5 implies that the locus of $\widetilde{K}_i(j\lambda, \omega)$ avoids the circle with center $-\sigma_i/\rho_i$ and radius $|\Delta_i/\rho_i|$ for almost every $\omega \epsilon \Omega$. A minimum $\alpha_i \epsilon R^+$ is sought so that this condition is met.

(b) If $\rho_i < 0$, Eq. 5.4 may be written, for almost every $\omega \epsilon \Omega$, as

$$0 \geq \frac{1}{\rho_i} + (\frac{\sigma_i}{\rho_i})\widetilde{K}_i + (\frac{\sigma_i}{\rho_i})\widetilde{K}_i^* + \widetilde{K}_i \widetilde{K}_i^* \ ,$$

and we proceed to Eq. 5.5 with the inequality reversed. This implies that we are seeking a minimum $\alpha_i \epsilon R^+$ such that the locus of $\widetilde{K}_i(j\lambda, \omega)$,

$\lambda \epsilon R$, is contained in a circle with center $- \sigma_i/\rho_i$ and radius $|\Delta_i/\rho_i|$ for almost every $\omega \epsilon \Omega$.

(c) If $\rho_i = 0$ (if ρ_i changes sign for $0 < \alpha_i < 1$, we need to consider the possibility of this case) we may write Eq. 5.4 as

$$0 \leq \sigma_i \widetilde{K}_i^* + \sigma_1 \widetilde{K}_1^* + 1,$$

or equivalently as

$$\sigma_i R_e (\widetilde{K}_i) + \frac{1}{2} \geq 0 \qquad \text{a.e.[P]}.$$

That is, we require

$$\mathrm{Re}\widetilde{K}_i(j\lambda, \omega) \geq - \frac{1}{2\sigma_i}, \qquad \lambda \epsilon R, \qquad \text{a.e.[P]}$$

(for ρ_i to change signs for $0 < \alpha_i < 1$, it is necessary that $\sigma_i > 0$).

Remark 5.5. Condition (ii) of Theorem 5.1 may be checked graphically if $N_i = 1$, by applying the principle of the argument (see, for instance, Holtzman [5]) for complex functions. To satisfy the inequality

$$1 + \frac{1}{2} (a_i + b_i)\widetilde{K}_i(s, \omega) \neq 0 \text{ for } \mathrm{Re}(s) \geq 0, \qquad \text{a.e.[P]}$$

we require that the locus of $\widetilde{K}_i(j\lambda, \omega)$, $\lambda \epsilon R$, does not encircle the point $(2/(a_i + b_i), 0)$ with probability one.

An example using these graphical techniques is worked in Chapter 6 (Example 6.5).

The following theorem is a composite stochastic system version of the Popov stability criterion.

<u>Theorem 5.2</u>. Continuous-time System 5.1 is stochastically absolutely stable if the following conditions hold:

(i) $N_i = 1$, $i \in M$;

(ii) $D_{ij} = 0$, $i, j \in M$, B_{ij} is a Type $\mathcal{B}$ operator with $\xi_{ij}(e_m(t, \omega)) = e_j(t, \omega)$, $i, j \in M$;

(iii) $\psi_i \in \mathcal{N}_{(1)}$ with $a_i = 0$, $b_i > 0$, $i \in M$;

(iv) $k_i, \dot{k}_i \in L_1(R^+, L_\infty(\Omega))$, $k_i \in L_2(R^+, L_\infty(\Omega))$, $i \in M$;

(v) $r_i, \dot{r}_i \in L_2(R^+, L_\infty(\Omega))$, $|r_i(t, \omega)| \to 0$ as $t \to \infty$, a.e.[P], $i \in M$;

(vi) there exists a $q_i > 0$ such that

$$\mathrm{Re}[(1 + j\lambda q_i)\widetilde{K}_i(j\lambda, \omega)] + b_i^{-1} \geq \delta_i > 0, \qquad \text{a.e.[P]}, \ \lambda \in R^+$$

for some real δ_i; and

(vii) the test matrix $A = [a_{ij}]$ has positive successive principal minors, where

$$a_{ij} = \begin{cases} 1 - (\beta_{ii} + \alpha_i \delta_i^{-1} \gamma_{ii}) & i = j \\ - \alpha_i \delta_i^{-1} \gamma_{ij} - \beta_{ij}) & i \neq j \quad i, j \ M \end{cases}$$

with

$$\alpha_i \geq \sup_{\lambda \in R^+} |\widetilde{K}_i(j\lambda, \omega)| \qquad \text{a.e.[P]},$$

$$\gamma_{ik} \geq \sup_{\lambda \in R^+} |(1 + j\lambda q_k)\widetilde{B}_{ik}(j\lambda, \omega)| \qquad \text{a.e.[P]},$$

$$\beta_{ik} \geq \sup_{\lambda \in R^+} |\widetilde{B}_{ik}(j\lambda, \omega)| \qquad \text{a.e.[P]}.$$

Remark 5.6. The comments of Remark 5.2 hold for Theorem 5.2 as well. For the deterministic case with $m = N_1 = 1$, Theorem 5.2 reduces to a version of the Popov-like theorems of Sandberg [35] and Zames [44, 45]. For $m = 1$, Theorem 5.2 is somewhat similar to Theorem 9.2.1 of Tsokos and Padgett [38]. We do not, however, require boundedness or continuity of the nonlinearity, ψ_i, as in [38].

Remark 5.7. Condition (vi) of Theorem 5.2 is the familiar Popov condition. The value of δ_i may be determined graphically. It is the minimum distance, parallel to the real axis, between the graph of the modified Nyquist plot of the linear operator K_i and the Popov line with intercept $- b_i^{-1}$ and slope q_i^{-1}.

Remark 5.8. In setting the D_{ij}, $i, j \in M$, terms of Eq. 5.1 to zero, we are allowing the subsystems to be interconnected only through the error terms, $e_i(t, \omega)$. This, however, is quite natural when applying the theorem to interconnected systems described by differential equations (see Section 5.4) and is rather flexible in control system work (see McClamroch [19]).

5.4 Applications to Nonlinear Differential Equations

In this section we present conditions for stability of interconnected stochastic systems governed by one of the following two types of differential equations,

$$\frac{dx_i(t, \omega)}{dt} = A_i(\omega)x_i(t, \omega) + \psi_i(x_i(t, \omega), t, \omega) + f_i(t, \omega) + \sum_{j=1}^{m} d_{ij}(\omega)x_j(t, \omega) \tag{5.6}$$

or

$$\frac{dx_i(t, \omega)}{dt} = A_i(\omega)x_i(t, \omega) + v_i(\omega)\psi_i(x_i(t, \omega), t, \omega) + f_i(t, \omega) + \sum_{j=1}^{m} d_{ij}(\omega)\sigma_i(t, \omega) \tag{5.7}$$

with

$$\sigma_i(t, \omega) = c_i^T(\omega)x_i(t, \omega),$$

where for both equations, $i, j \epsilon M = \{1, 2, \ldots m\}$. It is assumed that for Eq. 5.6 and 5.7 with $i, j \epsilon M$, $A_i(\omega)$ is an $N_i \times N_i$ matrix whose elements are F-measurable functions of ω; $x_i(t, \omega)$, $c_i(\omega)$, $v_i(\omega)$, and $f_i(t, \omega)$ are $N_i \times 1$ vectors whose elements are random variables for each $t \epsilon R^+$, and where the elements of $c_i(\omega)$ and $v_i(\omega)$ are essentially bounded; $d_{ij}(\omega)$ is an $N_i \times N_j$ random matrix; $\sigma_i(t, \omega)$ is a scalar random variable for each $t\ R^+$. For $M = \{1\}$, Eq. 5.7 is similar to one studied by Tsokos [37] and Tsokos and Padgett [38].

We apply Theorem 5.1 to determine conditions for stochastic absolute stability of systems governed by Eq. 5.6 and Theorem 5.2 to determine the stability of systems governed by Eq. 5.7.

<u>Theorem 5.3</u>. The differential System 5.6 is stochastically absolutely stable if the following conditions hold:

(i) $A_i(\omega)$ is a stochastically stable matrix, $i \epsilon M$;

(ii) $f_i \epsilon L_{2(N_i)}(R^+, L_\infty(\Omega))$, $i \epsilon M$;

(iii) $\psi_i \epsilon \eta_{(N_i)}$, $i \epsilon M$;

(iv) $\det [(s + \frac{1}{2}(a_i + b_i))I - A_i] \neq 0$ for $Re(s) \geq 0$, a.e.[P], $i \epsilon M$; and

(v) the test matrix $C = [c_{ij}]$ has positive successive principal minors, where

$$c_{ik} = \begin{cases} 1 - \gamma_i & i = k \\ -\delta_{ik} & i \neq k, \quad i, k\ M, \end{cases}$$

with

$$\gamma_i \geq \frac{1}{2}(b_i - a_i) \sup_{\lambda \epsilon R^+} \Lambda[(j\lambda + \frac{1}{2}(a_i + b_i))I + A_i(\omega))^{-1}]$$

a.e.[P], and

$$\delta_{ik} \geq \sup_{\lambda \epsilon R^+} [((j\lambda + \frac{1}{2}(a_i + b_i))I + A_i(\omega))^{-1}$$

$$\cdot (j\lambda I - A_i(\omega))d_{ij}(\omega)] \qquad \text{a.e.[P]}.$$

<u>Theorem 5.4</u>. The differential System 5.7 is stochastically absolutely stable if the following conditions hold:

(i) $\psi_i \epsilon \eta_{(1)}$ with $a_i = 0$, $b_i > 0$, $i \epsilon M$;

(ii) $A_i(\omega)$ is a stochastically stable matrix, $i \epsilon M$;

(iii) $f_i \epsilon L_2(R^+, L_\infty(\Omega))$, $i \epsilon M$;

(iv) there exists a $q_i > 0$ such that

$$Re[(1 + j\lambda q_i)c_i^T(\omega)(j\lambda I - A_i(\omega))^{-1}v_i(\omega)] + b_i^{-1} \geq \delta_i > 0,$$

a.e.[P], for some real δ_i; and

(v) the test matrix $C = [c_{ik}]$ has positive successive principal minors, where

$$c_{ik} = \begin{cases} 1 - \alpha_i \delta_i^{-1} \gamma_{ii} + \beta_{ii} & i = k \\ - \alpha_i \delta_i^{-1} \gamma_{ik} - \beta_{ik} & i \neq k, \quad i, k \in M, \end{cases}$$

with

$$\alpha_i \geq \sup_{\lambda \in R^+} |c_i^T(\omega)(j\lambda I - A_i(\omega))^{-1} v_i(\omega)| \qquad \text{a.e.[P]},$$

$$\gamma_{ik} \geq \sup_{\lambda \in R^+} |(1 + j\lambda q_i) c_i^T(\omega)(j\lambda - A_i(\omega))^{-1} v_i(\omega) d_{ik}(\omega)|$$

a.e.[P],

$$\beta_{ik} \geq \sup_{\lambda \in R^+} |c_i^T(\omega)(j\lambda I - A_i(\omega))^{-1} v_i(\omega) d_{ik}(\omega)| \qquad \text{a.e.[P]}.$$

6. EXAMPLES

6.1 Introduction

In this chapter we present some specific examples of control system stability and instability analysis using the techniques developed in the previous chapters. These examples were not necessarily formulated on the basis of some particular physical systems. They are used to demonstrate the utility and flexibility of the results of Chapters 4, 5, and 6. Examples 6.1, 6.2, and 6.3 demonstrate the application of material from Chapter 3. Examples 6.1 and 6.3 are stability examples, while Example 6.2 is an instability example. Example 6.4 represents an application of Corollary 4.1 using the frequency-domain graphical approach. Theorem 5.1 is used in Example 6.5. In this example the frequency-domain techniques of Remarks 5.1 and 5.4 are utilized. Due to similarity we do not present examples involving the application of all the theorems of previous chapters.

6.2 Examples

Example 6.1. Consider the discrete-time system shown in Fig. 6.1, consisting of three Type 2S Subsystems 3.4 (see Section 3.2). Subsystem 1 is described by

$$e_1(n, \omega) = r_1(n, \omega) + y_2(n, \omega) + (\nu_{13}(n) + \sqrt{2^{-1}})y_3(n, \omega) - (\sqrt{e^2 - 1} - 1) \sum_{\ell=0}^{n-1} e^{-(n-\ell)} f_1(\ell, \omega) e_1(\ell, \omega) \qquad (6.1)$$

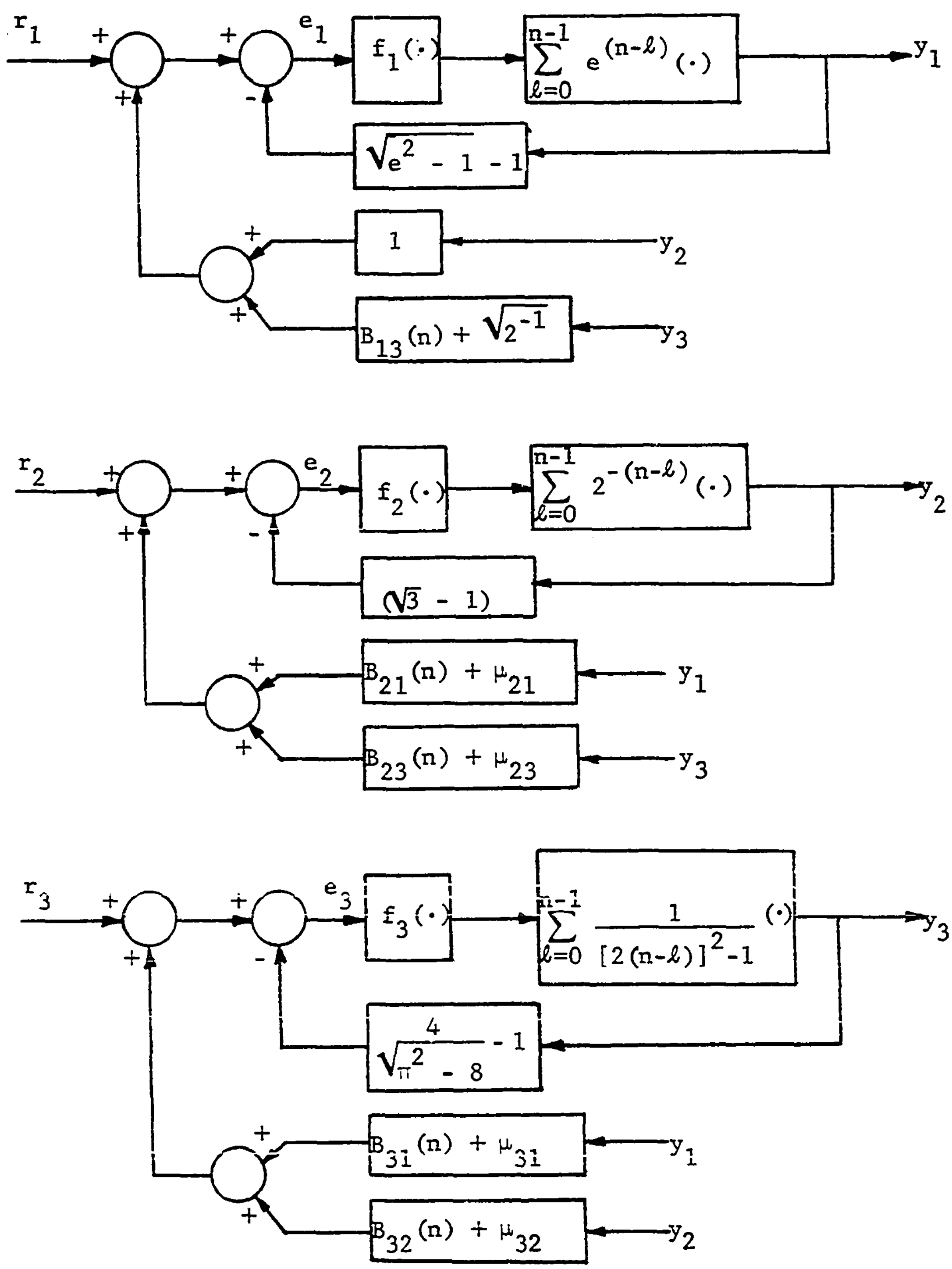

Fig. 6.1. Block diagram of the system for Example 6.1

where e, without arguments, represents the base of natural logarithms (2.718 ...). Subsystem 2 is described by

$$e_2(n, \omega) = r_2(n, \omega) + (\nu_{21}(n) + \mu_{21})y_1(n, \omega)$$
$$+ (\nu_{23}(n) + \mu_{23})y_3(n, \omega)$$
$$- (\sqrt{3} - 1) \sum_{\ell=0}^{n-1} 2^{-(n-\ell)} f_2(\ell, \omega)e_2(\ell, \omega); \qquad (6.2)$$

and Subsystem 3 is described by

$$e_3(n, \omega) = r_3(n, \omega) + (\nu_{31}(n) + \mu_{31})y_1(n, \omega)$$
$$+ (\nu_{32}(n) + \mu_{32})y_2(n, \omega)$$
$$- \left(\frac{4}{\sqrt{\pi^2 - 8}} - 1\right) \sum_{\ell=0}^{n-1} \frac{1}{4(n - \ell)^2 - 1} f_3(\ell, \omega)e_3(\ell, \omega), \qquad (6.3)$$

where $f_i(n, \omega)$, $i \epsilon M = \{1, 2, 3\}$, denotes a discrete noise process with statistics

$$Ef_i(n, \omega) = 0, \qquad n \epsilon I^+, \qquad i \epsilon M,$$

$$Ef_i(n, \omega)f_j(m, \omega) = \begin{cases} 0 & n \neq m \quad i, j\ M \\ 0 & n = m \quad i \neq j \\ 1 & n = m \quad i = j \quad n, m \epsilon I^+, \qquad i, j \epsilon M, \end{cases}$$

and $f_i(n, \omega)$ and $r_j(m, \omega)$ are independent for all i, j, n, and m. The processes $\nu_{ij}(n)$, $i, j \epsilon M$, $n \epsilon I^+$, represent sequences of second-order independent random variables such that

$$E\nu_{ij}(n) = 0, \qquad n \epsilon I^+, \qquad i, j \epsilon M, \text{ and}$$

$$E\nu_{ij}(n)\nu_{ij}(m) = \begin{cases} 0 & n \neq m \\ \sigma_{ij}^2 & n = m, \qquad n, m \epsilon I^+, \qquad i, j \epsilon M, \end{cases}$$

where σ_{ij}^2 R^+. Also the variables μ_{ij} are real scalars, representing bias terms, added to the corresponding processes, $\nu_{ij}(n)$. Note that $\mu_{12} = 1$ and $\mu_{13} = \sqrt{2^{-1}}$.

By comparing Eqs. 6.1-6.3 with System 3.3, the following identifications may be made, for $n \epsilon I^+$:

$$C_1 y_1(n, \omega) = (\sqrt{e^2 - 1} - 1) y_1(n, \omega)$$

$$H_1 e_1(n, \omega) = \sum_{\ell=0}^{n-1} e^{-(n-\ell)} f_1(\ell, \omega) e_1(\ell, \omega)$$

$$B_{11} y_1(n, \omega) = 0$$

$$B_{12} y_2(n, \omega) = y_2(n, \omega)$$

$$B_{13} y_3(n, \omega) = (\nu_{13}(n) + \sqrt{2^{-1}}) y_3(n, \omega)$$

$$C_2 y_2(n, \omega) = (\sqrt{3} - 1) y_2(n, \omega)$$

$$H_2 e_2(n, \omega) = \sum_{\ell=0}^{n-1} 2^{-(n-\ell)} f_2(\ell, \omega) e_2(\ell, \omega)$$

$$B_{21} y_1(n, \omega) = (\nu_{21}(n) + \mu_{21}) y_1(n, \omega)$$

$$B_{22} y_2(n, \omega) = 0$$

$$B_{23} y_3(n, \omega) = (\nu_{23}(n) + \mu_{23}) y_3(n, \omega)$$

$$C_3y_3(n, \omega) = \left(\frac{4}{\sqrt{\pi^2 - 8}} - 1\right)y_3(n, \omega)$$

$$H_3e_3(n, \omega) = \sum_{\ell=0}^{n-1} \frac{1}{4(n - \ell)^2 - 1} f_3(\ell, \omega)e_3(\ell, \omega)$$

$$B_{31}y_1(n, \omega) = (\nu_{31}(n) + \mu_{31})y_1(n, \omega)$$

$$B_{32}y_2(n, \omega) = (\nu_{32}(n) + \mu_{32})y_2(n, \omega)$$

$$B_{33}y_3(n, \omega) = 0$$

From Definition 3.4 it can be seen that our initial statement that the three subsystems are of Type 2S is indeed correct.

In this example, the operators B_{ij}, $i \neq j$, represent uncertainties in the interconnecting structure (with the exception of B_{12}, which represents the identity operator) modeled by a constant or bias term μ_{ij}, plus white noise, $\nu_{ij}(n)$. With the constraints that

$$\sigma_{12}^2 + \mu_{12}^2 = \sigma_{23}^2 + \mu_{23}^2 \triangleq \psi_2^2, \text{ and}$$

$$\sigma_{31}^2 + \mu_{31}^2 = \sigma_{32}^2 + \mu_{32}^2 \triangleq \psi_3^2,$$

we determine the range over which ψ_2 and ψ_3 may vary and still guarantee the second-order stochastic input-output stability of the system.

Since

$$y_i(n, \omega) = \sum_{\ell=0}^{n-1} w_i(n - \ell)f_i(\ell, \omega)e_i(\ell, \omega)$$

$y_i(n, \omega)$ depends only on $e_i(\ell, \omega)$ and $f_i(\ell, \omega)$ for $\ell < n$ and not on $e_i(n, \omega)$ or $f_i(n, \omega)$. Hence $y_j(n, \omega)$ and $\nu_{ij}(n)$ are stochastically independent so that

$$E(B_{ij}y_j(n, \omega))^2 = E\nu_{ij}^2(n)Ey_j^2(n, \omega) + \mu_{ij}^2 Ey_j^2(n),$$

and hence

$$\|B_{ij}y_j(n, \omega)\|_N = (\sigma_{ij}^2 + \mu_{ij}^2)^{1/2}\|y_j(n, \omega)\|_N = \psi_i\|(y_j(n, \omega)\|_N$$

for $i \in \{2, 3\}$, $j \in M$, $n, N \in I^+$. In the notation of Theorem 3.2 we have

$$\left.\begin{aligned}
d_{11} &= d_{22} = d_{33} = 0 \\
d_{12} &= d_{13} = 1 \\
d_{21} &= d_{23} = \psi_2 \\
d_{31} &= d_{32} = \psi_3 \\
g_1 &= (\sqrt{e^2 - 1} - 1) \\
g_2 &= (\sqrt{3} - 1) \\
g_3 &= \frac{4}{\sqrt{\pi^2 - 8}} - 1
\end{aligned}\right\} \qquad (6.4)$$

Note that

$$[\sum_{\ell=0}^{n-1} w_1^2(n - \ell)\sigma_1^2]^{1/2} \le [\sum_{k=1}^{\infty} e^{-2k}]^{1/2} = (\frac{1}{e^2 - 1})^{1/2} ;$$

$$[\sum_{\ell=0}^{n-1} w_2^2(n - \ell)\sigma_2^2]^{1/2} \le [\sum_{k=1}^{\infty} 2^{-2k}]^{1/2} = (\frac{1}{3})^{1/2} ; \text{ and}$$

$$[\sum_{\ell=0}^{n-1} w_3^2(n - \ell)\sigma_3^2]^{1/2} \le [\sum_{k=1}^{\infty} (\frac{1}{4k^2 - 1})^2]^{1/2} = (\frac{\pi^2 - 8}{16})^{1/2} .$$

Hence we may choose $\alpha_1 = 1/\sqrt{e^2 - 1}$, $\alpha_2 = 3^{-1/2}$, $\alpha_3 = ((\pi^2 - 8/16)^{1/2}$.
The test-matrix, A, of Theorem 3.2 assumes the form

$$A = \begin{bmatrix} 1 - \alpha_1 g_1 & - \alpha_2 d_{12} & - \alpha_3 d_{13} \\ - \alpha_1 d_{21} & 1 - \alpha_2 g_2 & - \alpha_3 d_{23} \\ - \alpha_1 d_{31} & - \alpha_2 d_{32} & 1 - \alpha_3 g_3 \end{bmatrix}.$$

Note that the successive principal minors of A have the same sign as the principal diagonal minors of

$$A' = \begin{bmatrix} \frac{1 - \alpha_1 g_1}{\alpha_1} & - d_{12} & - d_{13} \\ - d_{21} & \frac{1 - \alpha_2 g_2}{\alpha_2} & - d_{23} \\ - d_{31} & - d_{32} & \frac{1 - \alpha_3 g_3}{\alpha_3} \end{bmatrix} \quad (6.5)$$

and hence, for our purposes, A and A' are equivalent. Also, we have $g_i = 1/\alpha_i - 1$, $i \epsilon M$, and thus

$$\frac{1 - \alpha_i g_i}{\alpha_i} = \frac{1 - \alpha_i (\frac{1}{\alpha_i} - 1)}{\alpha_i} = 1, \qquad i \epsilon M.$$

Therefore our modified A-matrix becomes

$$A' = \begin{bmatrix} 1 & - 1 & - 1 \\ - \psi_2 & 1 & - \psi_2 \\ - \psi_3 & - \psi_3 & 1 \end{bmatrix}.$$

The matrix A' (and therefore the matrix A) has positive successive principal minors if and only if

$$1 > 0 \tag{6.6a}$$

$$1 - \psi_2 > 0 \tag{6.6b}$$

$$1 - (4\psi_2\psi_3 + \psi_2 + \psi_3) > 0. \tag{6.6c}$$

Note that Inequalities 6.6a-6.6c hold if 6.6c does. It now follows from Theorem 3.2 that the system of Eqs. 6.1-6.3 and Fig. 6.1 is second-order stochastic input-output stable if $\psi_2 > 0$, $\psi_3 > 0$ and Inequality 6.6c is satisfied. The region of stability in the $\psi_1 - \psi_2$ plane represented by Inequality 6.6c is depicted in Fig. 6.2.

Example 6.2. For this example we will use the system of Eqs. 6.1-6.3 and Fig. 6.1, with the modifications

$$B_{ij}y_j(n, \omega) = y_j(n, \omega), \qquad i, j\ M,\ n\epsilon I^+,$$

that is, we replace the random interconnection coefficients of the form $\nu_{ij}(n) + \mu_{ij}$ with a unity multiplier. We also allow the feedback operator, C_1, to be of the form

$$C_1y_1(n, \omega) = \overline{C}_1y_1(n, \omega), \qquad n\epsilon I^+,$$

where $\overline{C}_1 \epsilon R^+$. We will use Theorem 3.6 to determine over what range of values of $\overline{C}_1 \epsilon R^+$ we are guaranteed that the modified system is second-order stochastic input-output unstable. In order to accomplish this, we will determine over what range of $\overline{C}_1 \epsilon R^+$ condition (iv) of Theorem 3.6 holds; that is, we will determine over what range of $\overline{C}_1 \epsilon R^+$ the inequality

$$(\bar{C}_1\sigma_1)^2 \sum_{\ell=1}^{\infty} w_1^2(\ell) \geq 1 \qquad (6.7)$$

holds. Since $\sigma_1^2 = 1$ and

$$\sum_{\ell=1}^{\infty} w_1^2(\ell) = \sum_{\ell=1}^{\infty} e^{-2\ell} = \frac{1}{e^2 - 1} ,$$

it follows that Inequality 6.7 holds for

$$\bar{C}_1 \geq \sqrt{e^2 - 1} . \qquad (6.8)$$

Since, by the definition of the example, the remaining conditions of Theorem 3.6 hold for the range of $\bar{C}_1 \epsilon R^+$ given by Inequality 6.8, the modified system is second-order stochastic input-output unstable.

As a comparison, we use Theorem 3.2 to determine the range of $\bar{C}_1 \epsilon R^+$ for which second-order stochastic input-output stability for the modified system can be guaranteed. For the modified system, the identifications of Eq. 6.4 hold with the exceptions

$$d_{21} = d_{23} = d_{31} = d_{32} = 1, \text{ and}$$

$$g_1 = \bar{C}_1 .$$

In order to determine the stability range of $\bar{C} \epsilon R^+$, we use the alternate form of the test-matrix, A', given by Eq. 6.5. In this case we have

$$A' = \begin{bmatrix} \frac{1 - \alpha_1\bar{C}_1}{\alpha_1} & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} .$$

Note that det A' = - 4, regardless of the values of α_1 and $\bar{C}_1$. This indicates that the values of the multipliers in the interconnecting structure (the operators B_{ij}) are too large (unity). As one might expect, system stability depends on the values of the gains in the interconnecting structure (d_{ij}), even though we can predict system instability on the basis of subsystem instability, independent of the interconnecting structure gains. In order to observe this relationship, for the sake of the example, we choose the interconnecting operators, B_{ij}, to be of the form of Eq. 3.7, that is

$$B_{ij}y_j(n, \omega) = b \cdot y_j(n, \omega), \qquad b \epsilon R^+, \qquad i, j \epsilon M, \qquad n \epsilon I^+.$$

In this case

$$d_{ij} = b, \qquad i, j \epsilon M,$$

and the modified test-matrix, A', as given by Eq. 6.5 becomes

$$A' = \begin{bmatrix} \frac{1 - \alpha_1 \bar{C}_1}{\alpha_1} & - b & - b \\ - b & 1 & - b \\ - b & - b & 1 \end{bmatrix}.$$

The requirements for A' to have positive successive principal minors are

$$\frac{1 - \alpha_1 \bar{C}_1}{\alpha_i} > 0,$$

$$\frac{1 - \alpha_1 \bar{C}_1}{\alpha_1} - b^2 > 0, \text{ and}$$

$$- (1 + \frac{1}{b}) - (\frac{1 - \alpha_1 \bar{C}_1}{\alpha_1 b} + 1) + \frac{1}{b^3} (\frac{1 - \alpha_1 \bar{C}_1}{\alpha_1} - b^2) > 0. \qquad (6.9)$$

It can be seen that if Inequality 6.9 holds, the other two do also. Equation 6.9 may be reduced to

$$\bar{C}_1 < \frac{1}{\alpha_1} - 2b\left(\frac{b+1}{1-b^2}\right). \tag{6.10}$$

Recall that the choice of the interconnecting structure gain, b, has no effect on the instability results at the beginning of this example. That is, Inequality 6.8 still provides the instability region for the modified system. The regions of the b - $\bar{C}_1$ plane represented by Inequalities 6.8 and 6.10 are shown in Fig. 6.3. From this figure it can be seen that there exists a region in which we are not able to predict either stability or instability.

Example 6.3. Consider the continuous-time system shown in Fig. 6.4. It consists of three subsystems, each of a different type. Subsystem 1 is of Type 1S and is given by

$$e_1(t, \omega) = r_1(t, \omega) + 0.5y_2(t, \omega) + 0.3y_3(t, \omega) - y_1(t, \omega)$$

$$y_1(t, \omega) = \int_0^t e^{-2(t-s)} e_1(s, \omega) d\beta_1(s).$$

Subsystem 2 is of Type 3 and is given by

$$e_2(t, \omega) = r_2(t, \omega) + 1.2y_1(t, \omega) + y_3(t, \omega) - \frac{16}{9} y_2(t, \omega)$$

$$y_2(t, \omega) = \int_0^t e^{-3.5(t-s)} e_2(s, \omega) d\beta_2(s).$$

Subsystem 3 is of Type 5 and is given by

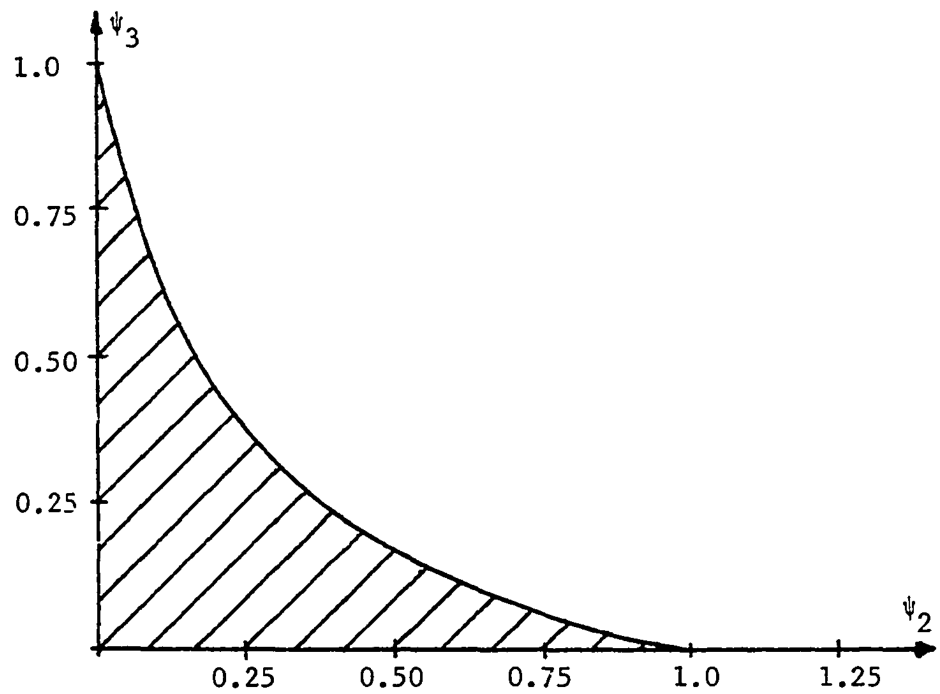

Fig. 6.2. Stability region for Example 6.1

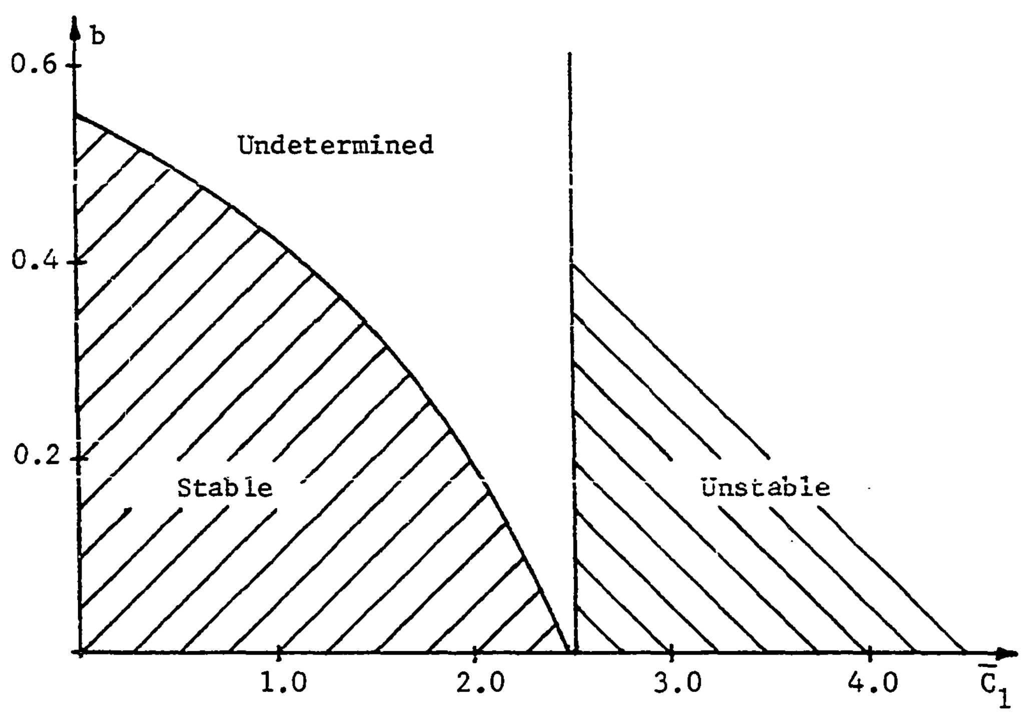

Fig. 6.3. Stability, instability, and indeterminate regions for Example 6.2

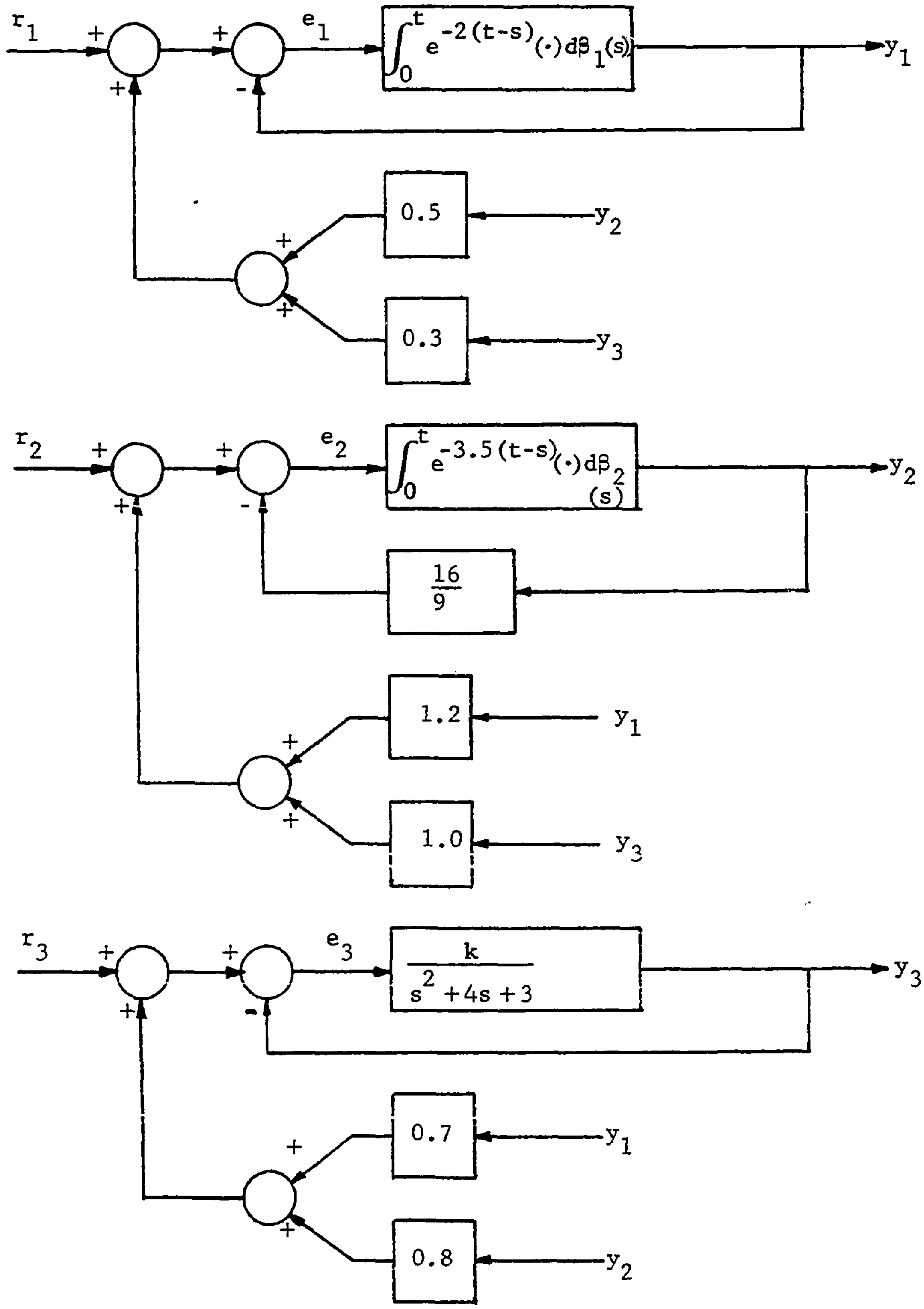

Fig. 6.4. Block diagram of the system for Example 6.3

$$e_3(t, \omega) = r_3(t, \omega) + 0.7y_1(t, \omega) + 0.8y_2(t, \omega) - y_3(t, \omega),$$

where the relationship between $y_3(t, \omega)$ and $e_3(t, \omega)$ is given in terms of the Laplace transforms of these two variables, $\tilde{y}_3(s, \omega)$ and $\tilde{e}_3(s, \omega)$, respectively,

$$\tilde{y}_3(s, \omega) = \frac{k}{s^2 + 4s + 3} \tilde{e}_3(s, \omega)$$

and where $k \epsilon R^+$. We wish to determine over what range of values of $k \epsilon R^+$ we are guaranteed second-order stochastic input-output stability of the system. The statistics of the Wiener processes, $\beta_i(s)$, are given by

$$E\beta_1(t) = E[\beta_2(t) - 1] = 0, \; t \epsilon R^+, \text{ and}$$

$$E[d\beta_1(t)] = E[d\beta_2(t) - t]^2 = dt, \; t \epsilon R^+.$$

We also assume that $\beta_i(t_1)$ and $r_j(t_2, \omega)$ are stochastically independent for all $i, j \epsilon M$ and $t_1, t_2 \epsilon R^+$.

Upon comparing the system of Fig. 6.4 to Eq. 3.1, the following identifications may be made for $t \epsilon R^+$:

$$C_1 y_1(t, \omega) = y_1(t, \omega)$$

$$H_1 e_1(t, \omega) = \int_0^t e^{-2(t-s)} e_1(s, \omega) d\beta_1(s)$$

$$B_{11} y_1(t, \omega) = 0$$

$$B_{12} y_2(t, \omega) = 0.5 y_2(t, \omega)$$

$$B_{13} y_3(t, \omega) = 0.3 y_3(t, \omega)$$

$$C_2y_2(t, \omega) = \frac{16}{9} y_2(t, \omega)$$

$$H_2e_2(t, \omega) = \int_0^t e^{-3.5(t-s)} e_2(s, \omega) d\beta_2(s)$$

$$B_{21}y_1(t, \omega) = 1.2y_1(t, \omega)$$

$$B_{22}y_2(t, \omega) = 0$$

$$B_{23}y_3(t, \omega) = y_3(t, \omega)$$

$$C_3y_3(t, \omega) = y_3(t, \omega)$$

$$B_{31}y_1(t, \omega) = 0.7y_1(t, \omega)$$

$$B_{32}y_2(t, \omega) = 0.8y_2(t, \omega)$$

$$B_{33}y_3(t, \omega) = 0$$

$$\tilde{h}_3(s) = \frac{k}{s^2 + 4s + 3}$$

where $\tilde{h}_3(s)$ is as given in Definition 3.6. From this list it is easy to compute the following terms of the test-matrix of Theorem 3.4; $d_{11} = d_{22} = d_{33} = 0$, $g_1 = g_3 = d_{23} = 1$, $g_2 = 16/9$, $d_{12} = 0.5$, $d_{13} = 0.3$, $d_{21} = 1.2$, $d_{31} = 0.7$, and $d_{32} = 0.8$. For $i = 1, 3$, $\gamma_i = 1$ by conditions (iii) and (iv) of Theorem 3.4. To determine α_1 note that

$$\left(\int_0^t w_1^2(t - s)\sigma_1^2(s)ds\right)^{1/2} \leq \left(\int_0^\infty e^{-4t}dt\right)^{1/2} = \frac{1}{2},$$

so that we may choose $\alpha_1 = \frac{1}{2}$. For Subsystem 2, we compute the Laplace transform of the resolvent corresponding to $w_2(t)$ (see Section 2.1)

$$\tilde{r}_2(s) = \frac{\tilde{w}_2(s)}{1 + \tilde{w}_2(s)} = \frac{(\frac{1}{s + 3.5})}{1 + (\frac{1}{s + 3.5})} = \frac{1}{s + 4.5}$$

and

$$r_2(t) = e^{-4.5t} .$$

Note that

$$[\frac{\sigma_2^2}{2\pi}\int_{-\infty}^{\infty} |\frac{\tilde{w}_2(j\lambda)}{1 + \tilde{w}_2(j\lambda)}|^2 \, d\lambda]^{1/2} = [\int_0^{\infty} r_2^2(t)dt]^{1/2}$$

$$= (\int_0^{\infty} e^{-9t}dt)^{1/2} = \frac{1}{3} .$$

Hence, we may choose $\alpha_2 = \frac{1}{3}$. Also,

$$1 + \int_0^{\infty} |r_2(t)|dt = 1 + \int_0^{t} e^{-4.5t}dt = \frac{11}{9} ,$$

and we may choose $\gamma_2 = \frac{11}{9}$. For Sybsystem 3 we have

$$h_3(t) = k(0.5e^{-t} - 0.5e^{-3t}) \geq 0$$

and we choose

$$\alpha_3 = \lim_{s\to 0} |h_i(s)| = \lim_{s\to 0} \frac{k}{s^2 + 4s + 3} = \frac{k}{3} ,$$

by Remark 3.5. The A-matrix of condition (vii) of Theorem 3.4 may now be written as

$$A = \begin{bmatrix} 1 - \alpha_1 g_1 & - \gamma_1\alpha_2 \, {}_{12} & - \gamma_1\alpha_3 d_{13} \\ - \gamma_2\alpha_2 d_{21} & 1 - \alpha_2 g_2 & - \gamma_2\alpha_3 d_{23} \\ - \gamma_3\alpha_1 d_{31} & - \gamma_3\alpha_2 d_{32} & 1 - \alpha_3 g_3 \end{bmatrix} .$$

As in Example 6.1, we use the equivalent test matrix

$$A' = \begin{bmatrix} \dfrac{1 - \alpha_1 g_1}{\alpha_1 \gamma_1} & - d_{12} & - d_{13} \\ - d_{21} & \dfrac{1 - \alpha_2 g_2}{\alpha_2 \gamma_2} & - d_{23} \\ - d_{31} & - d_{32} & \dfrac{1 - \alpha_3 g_3}{\alpha_3 \gamma_3} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & - 0.5 & - 0.3 \\ - 1.2 & 1 & - 1 \\ - 0.7 & - 0.8 & \frac{3}{k} - 1 \end{bmatrix}.$$

For A' (and hence A) to have positive successive principal minors, we require

$$1 > 0,$$

$$1 - 0.6 > 0, \text{ and}$$

$$0.4(\frac{3}{k} - 1) - 1.6480 > 0.$$

It follows from Theorem 3.4 that the system of Fig. 6.4 is second-order stochastic input-output stable if

$$0 \leq k \leq 0.586$$

(to three significant figures).

Example 6.4. Consider the system shown in Fig. 6.5. This system is governed by

$$e(t, \omega) = u(t, \omega) - \int_0^t g(t - s)\psi(e(s, \omega), s)d\beta(s)$$

where $\beta(s)$ is a Wiener process with

$$E\beta(t) = 0, \qquad t \in R^+,$$

$$E[d\beta(t)]^2 = \sigma^2 dt.$$

The convolution kernel $g(t)$, is assumed to possess the Laplace transform

$$\tilde{g}(s) = \frac{s + 2}{(s + 5)(s + 6)} .$$

We also assume that the nonlinearity, ψ, satisfies

$$\frac{1}{\sqrt{2}} \le \frac{\psi(x, t)}{x} \le 1, \qquad t \in R^+, \qquad x \in R.$$

We apply Corollary 4.1 to determine how large σ may become and still assume the second-order stochastic absolute input-output stability of the system.

Because $\tilde{g}(s)$ is a rational function with poles having positive real parts, we are assured that $g^2 \in L_1(R^+) \cap L_2(R^+)$. By Remark 4.1 we have

$$\tilde{g}_2(s) = \frac{-3(s + 7)}{(s + 10)(s + 11)} + \frac{4(s + 8)}{(s + 11)(s + 12)}$$

$$= \frac{s^2 + 15s + 68}{s^3 + 43s^2 + 362s + 1320}$$

and the locus of $\tilde{g}_2(j\lambda)$, $\lambda \in R^+$, is as shown in Fig. 6.6. From condition (ii) of Corollary 4.1, we compute the center of the circle as $(3/2\sigma^2, 0)$ and the radius as $1/2\sigma^2$. This places the left-most crossing of the real axis by the circle at $1/2\sigma^2$. From Fig. 6.6 it can be seen that if

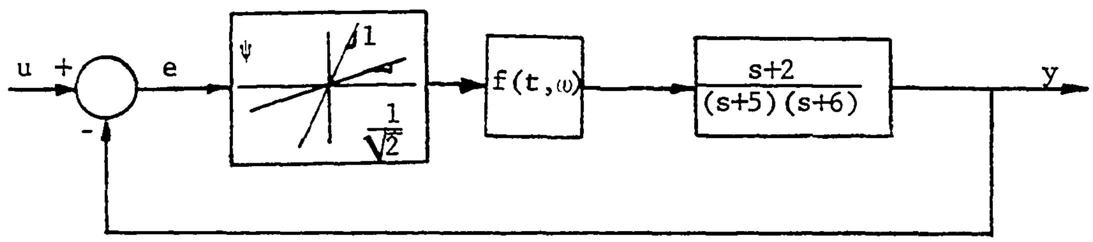

Fig. 6.5. Block diagram of the system for Example 6.4

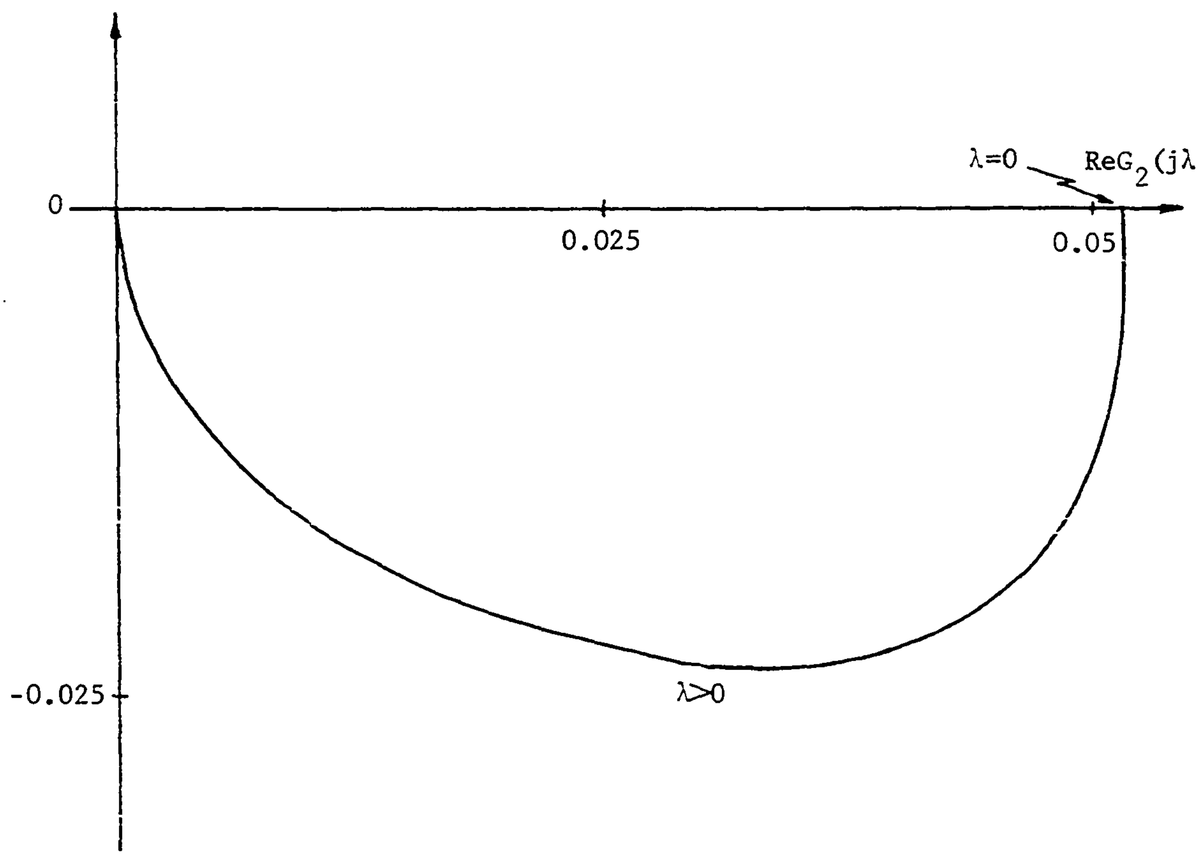

Fig. 6.6. Nyquist plot of $G_2(j\lambda)$ from Example 6.4

$1/\sigma^2 > 0.052$ (approximately), that is, if $\sigma < 19.2$, then the circle requirement, condition (ii) of Corollary 4.1, is satisfied and we conclude that for these values of σ the system shown in Fig. 6.5 is second-order stochastically absolutely input-output stable.

Example 6.5. We next apply Theorem 5.1 to the control system shown in Fig. 6.7, which is described functionally by

$$e_1(t, \omega) = r_1(t, \omega) - F_1 y_1(t, \omega) + y_2(t, \omega)$$

$$y_1(t, \omega) = \psi_1 N_1 e_1(t, \omega)$$

$$e_2(t, \omega) = - F_2 y_2(t, \omega) + y_1(t, \omega) + S_3 y_3(t, \omega)$$

$$y_2(t, \omega) = \psi_2 N_2 e_2(t, \omega)$$

$$e_3(t, \omega) = r_3(t, \omega) - y_3(t, \omega) + y_1(t, \omega)$$

$$y_3(t, \omega) = N_3 \psi_3 e_3(t, \omega),$$

where N_1, N_2, N_3, F_2, and S_3 are random convolution operators on $\mathcal{R}_2(R^+, L_\infty(\Omega))$, characterized by their transforms:

$$\tilde{N}_1(s, \omega) = \frac{G_1(s + 2)}{(s + 1)(s + 3)}$$

$$\tilde{N}_2(s, \omega) = \frac{10}{(s + 2)(s + 5)}$$

$$\tilde{N}_3(s, \omega) = \frac{s + 5}{(s + 6)(s + 2)}$$

$$\tilde{F}_2(s, \omega) = \frac{1}{s + 1}$$

$$\tilde{S}_3(s, \omega) = \frac{1}{s + d(\omega)}, \quad P[\omega\colon 4 \leq d(\omega) \leq 6] = 1;$$

F_1 is a Type A operator (see Section 5.2) with

$$F_1 y_1(t, \omega) = \theta(\omega) \cdot y_1(t, \omega),$$

with

$$P[\omega: \pi \leq \theta \leq 2\pi] = 1;$$

$\psi_1, \psi_2, \psi_3 \in N_{(1)}$ with

$$\psi_1 x_1(t, \omega) = \sin(x_1(t, \omega))$$

$$\psi_2 x_2(t, \omega) = a_1(\omega) x_2(t, \omega)$$

$$\psi_3 x_3(t, \omega) = 0.5t \sin(a_2(\omega) \cdot x_3(t, \omega)),$$

where $a_1(\omega)$ is a uniform random variable on $[0, 1]$, and $a_2(\omega)$ is a standard normal random variable; $e_1, e_2, e_3, y_1, y_2, y_3 \in E_2(R^+, L_\infty(\Omega))$ and $r_1, r_2, r_3 \in L_2(R^+, L_\infty(\Omega))$ (we will define $r_2(t, \omega) \equiv 0$). The gain, $G_1 \in R^+$, is to be determined in such a fashion as to insure a stochastically absolutely stable system. Note that in the present form, the system is not of the form of Eq. 5.1. In order to restructure the problem, define

$$e_1'(t, \omega) = N_1 e_1(t, \omega),$$

$$e_2'(t, \omega) = N_2 e_2(t, \omega),$$

$$r_1'(t, \omega) = N_1 r_1(t, \omega), \text{ and}$$

$$r_2'(t, \omega) = N_2 r_2(t, \omega).$$

We now have

$$N_1e_1(t, \omega) = N_1r_1(t, \omega) - N_1F_1\psi_1N_1e_1(t, \omega) + N_1y_2(t, \omega)$$

$$N_2e_2(t, \omega) = - N_2F_2\psi_2N_2e_2(t, \omega) + N_2y_1(t, \omega) + N_2S_3y_3(t, \omega)$$

$$e_3(t, \omega) = r_3(t, \omega) - y_3(t, \omega) + y_1(t, \omega)$$

$$y_1(t, \omega) = \psi_1e_1'(t, \omega)$$

$$y_2(t, \omega) = \psi_2e_2'(t, \omega)$$

$$y_3(t, \omega) = N_3\psi_3e_3(t, \omega).$$

We rewrite the first three equations above as

$$e_1'(t, \omega) = r_1'(t, \omega) - N_1F_1\psi_1e_1'(t, \omega) + N_1\psi_2e_2'(t, \omega)$$

$$e_2'(t, \omega) = r_2'(t, \omega) - N_2F_2\psi_2e_2'(t, \omega) + N_2\psi_1e_1'(t, \omega) + N_2S_3y_3(t, \omega)$$

and

$$e_3(t, \omega) = r_3(t, \omega) - N_3\psi_3e_3(t, \omega) + \psi_1e_1'(t, \omega).$$

The modified system is of the form of Eq. 5.1, and is depicted in Fig. 6.8. Note that if we show that

$$P\left\{\omega: \lim_{t\to\infty} e_i'(t, \omega) = 0\right\} = 1 \qquad i = 1, 2$$

then $s\tilde{e}_1'(s, \omega) \to 0$ as $s \to 0$. Also, we have

$$\lim_{s\to 0} se_i'(s, \omega) = \lim_{s\to 0} s\tilde{N}_i(s, \omega)e_i(s, \omega) = 0 \qquad \text{a.e.}[P]$$

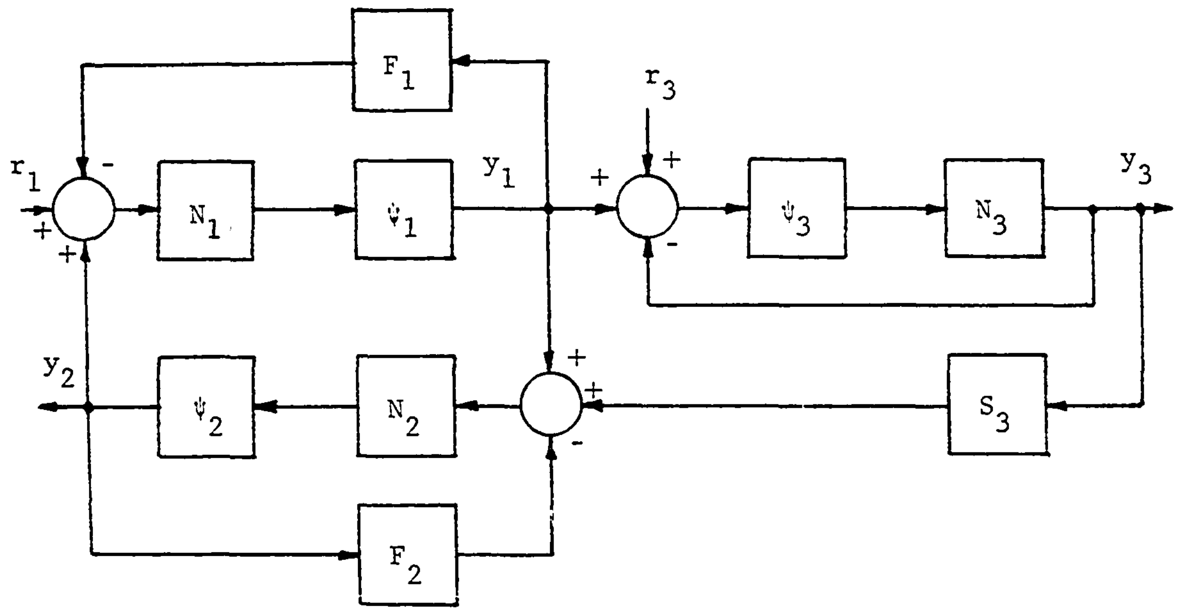

Fig. 6.7. Block diagram of the system for Example 6.5

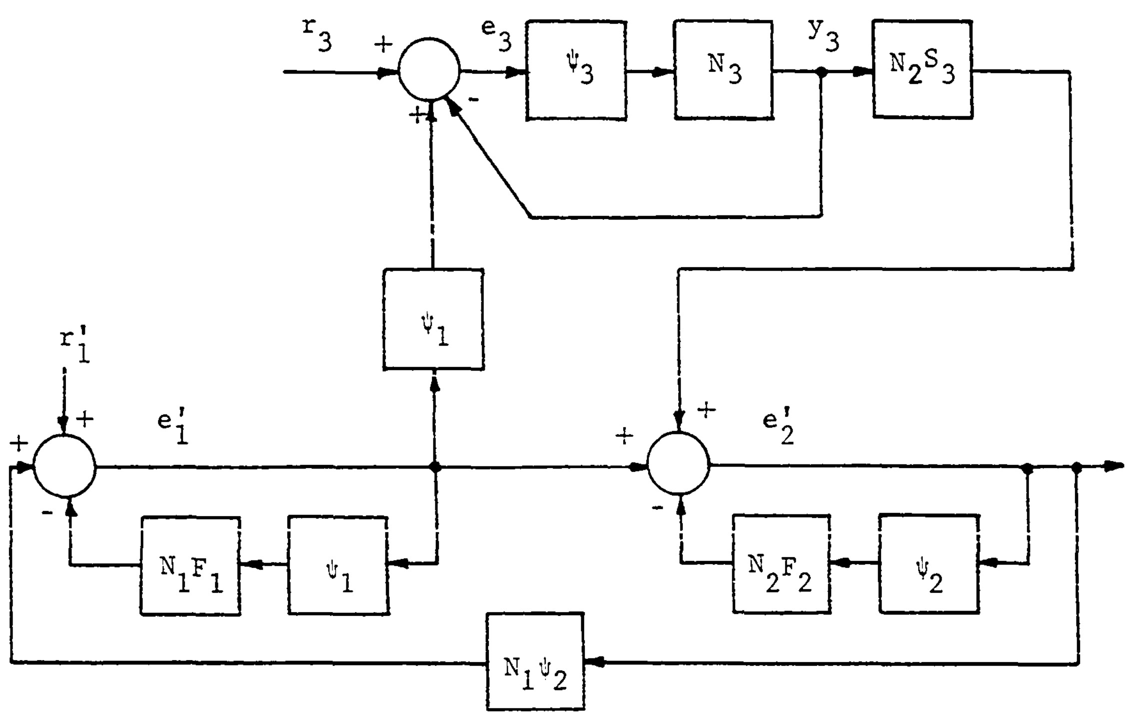

Fig. 6.8. Modified block diagram of the system for Example 6.5

Since $\lim_{s\to 0} \widetilde{N}_1(s, \omega) = 2G_1/3$ and $\lim_{s\to 0} \widetilde{N}_2(s, \omega) = 1$, it follows that $se_i(s, \omega) \to 0$ as $s \to 0$ a.e.[P]. Hence the two systems are equivalent for the purpose of determining stochastic absolute stability. By comparison with Eq. 5.1 we may make the following identifications: $K_1 = N_1F_1$, $K_2 = N_2F_2$, $K_3 = N_3$, $B_{12} = N_1\psi_2$, $B_{21} = N_2\psi_1$, $B_{31} = \psi_1$, $D_{23} = N_2S_3$, and $B_{11} = B_{13} = B_{22} = B_{23} = B_{32} = B_{33} = D_{11} = D_{12} = D_{13} = D_{21} = D_{22} = D_{31} = D_{32} = D_{33} = 0$. We check conditions (ii) and (iii) of Theorem 5.1 by the graphical method of Remark 5.4. The Nyquist plots of K_2 and K_3 are shown in Figs. 6.10 and 6.11. Note that since $K_1 = N_1F_1$ depends explicitly on ω through F_1, we only show a region where the locus of $\widetilde{K}_1(j\lambda, \omega)$ will fall with probability one. From the nonlinear elements, we determine the following parameters: $a_1 = -0.2122$, $b_1 = 1.0$, $a_2 = 0$, $b_2 = 1.0$, $a_3 = -0.5$, and $b_3 = 0.5$. We apply Remark 5.4 to determine the A-matrix terms α_i, $i = 1, 2, 3$. For Subsystem 1, we have from Eq. 5.3 that

$$\Delta_1 = \frac{b_1 - a_1}{2\alpha_1} = \frac{0.6061}{\alpha_1},$$

$$\sigma_1 = \frac{a_1 + b_1}{2} = 0.3939, \text{ and}$$

$$\rho_1 = \sigma_1^2 - \Delta_1^2 = 0.1552 - \frac{0.3674}{\alpha_1^2}.$$

Since $\rho_1 < 0$ for $0 < \alpha < 1$, we apply case (b) of Remark 5.4 and therefore we are looking for a circle that contains the locus of $\widetilde{K}_1(j\lambda, \omega)$, $\lambda \in R$, with probability one. The center and radius of allowable circles are given by c_1 and r_1, respectively, and may be computed from Remark 5.4:

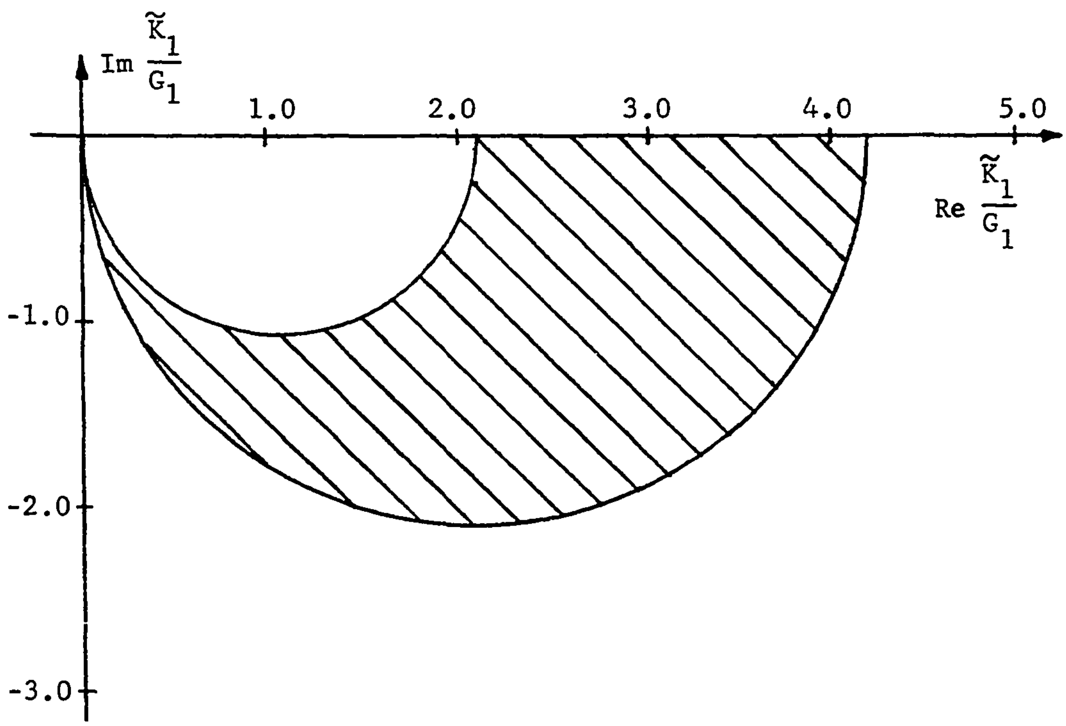

Fig. 6.9. Nyquist plot of $\frac{\tilde{K}_1(j\lambda,\ \omega)}{G_1}$ for Example 6.5

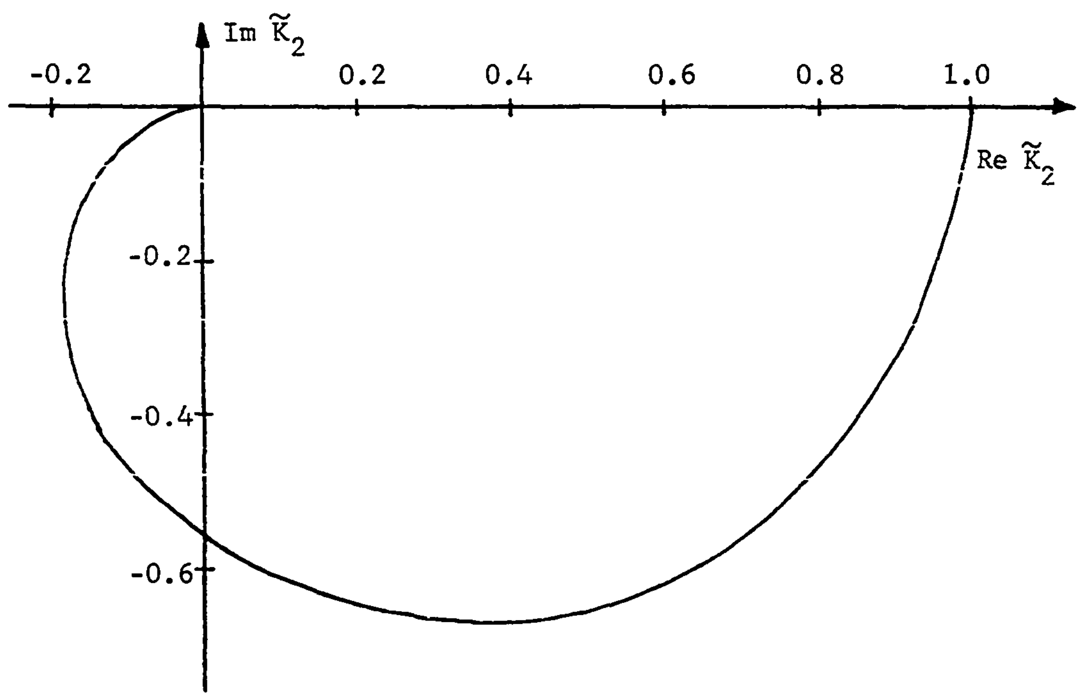

Fig. 6.10. Nyquist plot of $\tilde{K}_2(j\lambda)$ for Example 6.5

$$c_1 = -\frac{\sigma_1}{\rho_1} = \frac{0.3939\alpha_1^2}{0.3674 - 0.1552\alpha_1^2}, \text{ and}$$

$$r_1 = \left|\frac{\Delta_1}{\rho_1}\right| = \frac{0.6061\alpha_1}{0.3674 - 0.1552\alpha_1^2}.$$

Note that c_1 and r_1 are monotone increasing functions of α_1 for $0 < \alpha_1 < 1$. Using this fact and Fig. 6.9, we seek the minimum α_1 such that

$$c_1 + r_1 = \max_{\omega\in\Omega} \tilde{K}_1(\Gamma, \omega) = 4.1888G_1.$$

This is equivalent to the equation in α_1

$$\frac{0.3939\alpha_1^2 + 0.6061\alpha_1}{0.3674 - 0.1552\alpha_1^2} = 4.1888G_1.$$

Solving for α_1, we obtain

$$\alpha_1 = \frac{\sqrt{0.3674 + 2.4248G_1 + 4.0G_1^2} - 0.6061}{0.7878 + 13.002G_1} \tag{6.11}$$

For Subsystem 2 we have $\Delta_2 = 1/2\alpha_2$, $\sigma_2 = 0.5$, and $\rho_2 = 0.25(1 - \alpha_2^{-2})$. As in Subsystem 1, case (b) of Remark 5.4 applies since $\rho_2 < 0$ for $0 < \alpha_2 < 1$, and we are looking for a circle that contains the locus of $\tilde{K}_2(j\lambda, \omega)$, $\lambda\epsilon R$, with probability one. In this case the circle center and radius, c_2 and r_2, respectively, are given by

$$c_2 = \frac{2\alpha_2^2}{1 - \alpha_2^2}, \text{ and}$$

$$r_2 = \frac{2\alpha_2}{1 - \alpha_2^2}.$$

Again note that c_2 and r_2 are monotone increasing functions of α_2 and from Fig. 6.10, we are looking for an α_2 such that

$$c_2 + r_2 = \frac{2\alpha_2(\alpha_2 + 1)}{1 - \alpha_2^2} = 1.0.$$

In this case $\alpha_2 = 0.3333$ and the corresponding circle center and radius are given by $c_2 = 0.250$ and $r_2 = 0.750$, respectively. For Subsystem 3 we have $\Delta_3 = 1/2\alpha_3$, $\sigma_3 = 0$, $\rho_3 = -1/2\alpha_3$. Once again case (b) of Remark 5.4 applies, and we are looking for a circle that contains $K_3(j\lambda, \omega)$, $\lambda \epsilon R^+$, with probability one. In this case the circle center is the origin, (0, 0), and the radius is given by $2\alpha_3$. From Fig. 6.11 a circle with center (0, 0) and radius 5/12 will suffice. This corresponds to an α_3 of 0.2084. In order to determine the A-matrix coefficients γ_{ik}, we observe that

$$\|(1 + \frac{1}{2}(a_1 + b_1)K_1)^{-1}\frac{K_1}{\theta}\| \leq \frac{\alpha_1}{\theta} \leq \frac{\alpha_1}{\pi}.$$

So we choose $\gamma_{12} = \alpha_1/\pi$. Note that

$$\|(1 + \frac{1}{2}(a_2 + b_2)K_2)^{-1}B_{21}\| = \|(1 + \frac{1}{2}(a_2 + b_2)K_2)^{-1}N_2\psi_1\|$$

$$\leq \sup_{\lambda \epsilon R^+} \left|\frac{\tilde{N}_2(j\lambda, \omega)}{1 + \frac{1}{2}\tilde{K}_2(j\lambda, \omega)}\right|$$

$$= \sup_{\lambda \epsilon R^+} \left|\frac{10(j\lambda + 1)}{(j\lambda + 2)(j\lambda + 5)(j\lambda + 1) + 5}\right|$$

A plot of

$$\left|\frac{10(j\lambda + 1)}{(j\lambda + 2)(j\lambda + 5)(j\lambda + 1) + 5}\right| \quad \text{versus } \lambda$$

is shown in Fig. 6.12. It can be seen from this figure that the supremum over λ is somewhat less than 0.825. We use $\gamma_{21} = 0.825$ for demonstration purposes. For γ_{31} we have

$$\| (1 + \tfrac{1}{2}(a_3 + b_3)K_3)^{-1} B_{31} \| = \| \psi_1 \| \leq 1.0.$$

The choise $\gamma_{13} = 1.0$ is made. For the parameter ξ_{23}, we have

$$\| (1 + \tfrac{1}{2}(a_2 + b_2)K_2)^{-1} D_{23} \| = \| (1 + \tfrac{1}{2}(a_2 + b_2)K_2)^{-1} N_2 S_3 \|$$

$$\leq \| (1 + \tfrac{1}{2}(a_2 + b_2)K_2)^{-1} N_2 \| \cdot \| S_3 \| \leq \gamma_{21} \| S_3 \|$$

$$\leq \gamma_{21} \sup_{\lambda \in R^+} \left| \frac{1}{d + j\lambda} \right| = \frac{\gamma_{21}}{d} \leq 0.2063, \qquad \text{a.e.}[P].$$

We also have

$$\mu_2 = \max(0, 1) = 1.$$

The remaining A-matrix parameters are zero, $\gamma_{11} = \gamma_{22} = \gamma_{33} = \gamma_{13} = \gamma_{23} = \gamma_{32} = \xi_{11} = \xi_{12} = \xi_{13} = \xi_{21} = \xi_{22} = \xi_{31} = \xi_{32} = \xi_{33} = 0$. We compute the test-matrix, A, as

$$A = \begin{bmatrix} 1 - \alpha_1 & -\gamma_{12} & 0 \\ -\gamma_{21} & 1 - \alpha_2 & -\xi_{23} \\ -\gamma_{21} & 0 & 1 - \alpha_3 \end{bmatrix} = \begin{bmatrix} 1 - \alpha_1 & -\frac{\alpha_1}{\pi} & 0 \\ -0.825 & 0.6667 & -0.2063 \\ -1 & 0 & 0.7916 \end{bmatrix}$$

In order to satisfy condition (iii) of Theorem 5.1, we need positive successive principal minors of A, that is

$$1 - \alpha_1 > 0,$$

$$(1 - \alpha_1)(0.6667) - \frac{0.825\alpha_1}{\pi} > 0, \text{ and}$$

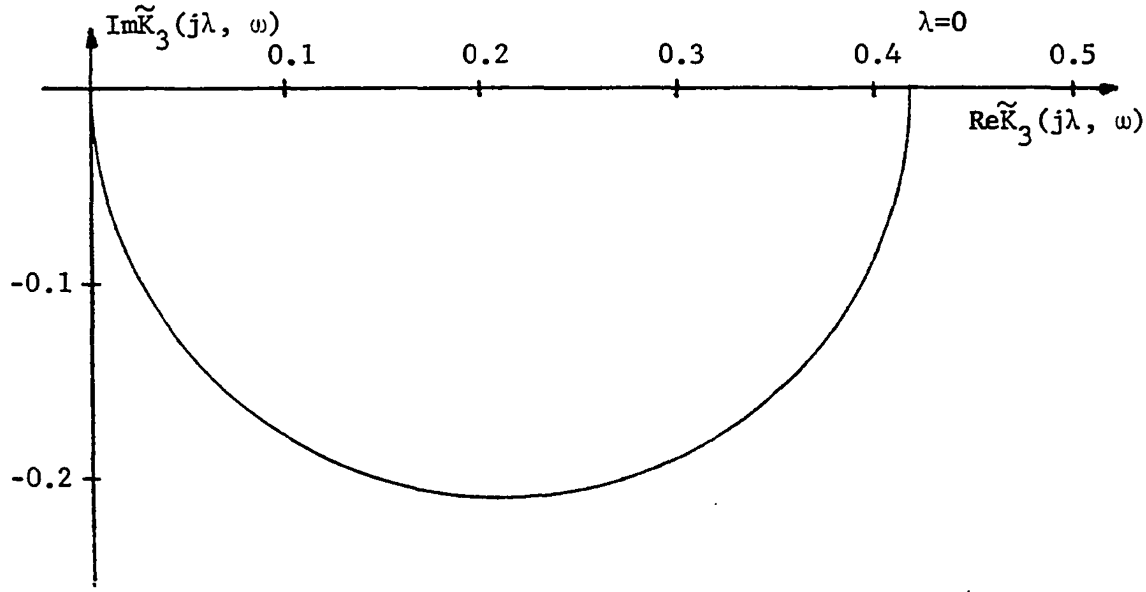

Fig. 6.11. Nyquist plot of $\tilde{K}_3(j\lambda,\ \omega)$ for Example 6.5

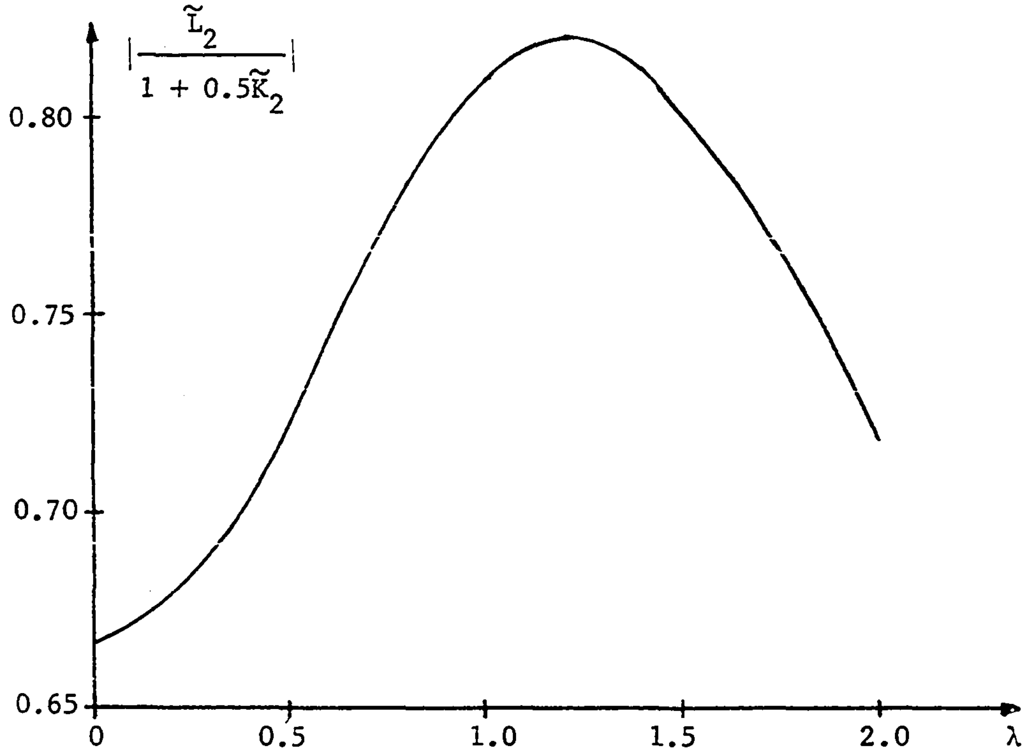

Fig. 6.12. Plot of $\left|\frac{\tilde{L}_2}{1 + 0.5\tilde{K}_2}\right|$ versus λ for Example 6.5

$$0.7916\ ((1 - \alpha_1)(0.6667) - \frac{0.825\alpha_1}{\pi}) - \frac{0.2063\alpha_1}{\pi} > 0.$$

It can be seen that all three inequalities are satisfied if the third one is. The third inequality is satisfied for $\alpha_1 < 0.6585$. In order to compute G_1, we must satisfy

$$\frac{\sqrt{0.3674 + 2.4248G_1 + 4.0G_1^2} - 0.6061}{0.7878 + 1.3002G_1} < 0.6585$$

or

$$0 < G_1 < 0.4535.$$

7. CONCLUDING REMARKS

7.1 Conclusions

New input-output stability results for large classes of multi input-multi output stochastic feedback systems have been established here. Whenever appropriate frequency domain interpretations were used. For the large-scale systems, the objective was always the same: to analyze composite systems in terms of lower order subsystems and in terms of the interconnecting structure. To demonstrate the methods of analysis advanced, several specific examples were considered.

7.2 Further Research

Many aspects of the stochastic system stability problem remain unsolved. The case where multiplicative gain is modeled as a constant plus white noise has been solved for linear systems [40], but remains an open question for nonlinear systems. When the gain term is modeled by multiplicative colored noise, the problem becomes more difficult. Martin and Johnson [17] and Willsky et al. [41] have results for certain restricted classes of linear systems, but in general the problem remains unsolved. No results currently exist for colored multiplicative noise in composite systems. As additional analytical tools are developed, more systems endowed with multiplicative noise can be handled correctly, instead of attempting to force them into an additive noise format.

One could use any of the techniques in this thesis or those referenced herein for design purposes, however, in general, the results tend to be somewhat conservative and the system designer is likely to turn to simulation to verify system stability. As more work is done in this area the results for specific types of systems tend to become less conservative.

As stated in the introduction to this thesis, some work has been done in the area of stochastic system stability but much more work lies ahead.

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10. APPENDIX A. PROOFS OF THEOREMS FROM CHAPTER 3

Proof of Theorem 3.1. For the ith Subsystem 3.2 we have $u_i, y_i \in S_{\infty e}$, so that

$$\|e_i(t)\|_T \leq \|u_i(t)\|_T + g_i \left\| \int_0^t w_i(t, s) e_i(s) d\beta_i(s) \right\|_T ,$$

where in the above inequality, as well as throughout the appendices, the explicit ω-dependence for the various processes is frequently suppressed. Noting that

$$Ey_i^2(t) = E\left[\int_0^t w_i(t, s) e_i(s) d\beta_i(s)\right]^2$$

$$= \int_0^t w_i^2(t, s)\sigma_i^2(s) Ee_i^2(s) ds$$

$$\leq \sup_{0\leq\tau\leq t} Ee_i^2(\tau) \cdot \int_0^t w_i^2(t, s)\sigma_i^2(s) ds$$

it follows from the definition of α_i and $\|\cdot\|_T$ that

$$\left\| \int_0^t w_i(t, s) e_i(s) d\beta_i(s) \right\|_T = \|y_i(t)\|_T \leq \alpha_i \|e_i(t)\|_T$$

and hence

$$\|e_i(t)\|_T \leq \|u_i(t)\|_T + g_i\alpha_i \|e_i(t)\|_T, \quad T \in R^+.$$

We also have

$$\|u_i(t)\|_T \leq \|r_i(t)\|_T + \sum_{j=1}^{m} d_{ij}\|y_j(t)\|_T \leq \|r_i(t)\|_T + \sum_{j=1}^{m} d_{ij}\alpha_j\|e_j(t)\|_T.$$

Using the vector notation $\|e(t)\|_T \triangleq [\|e_1(t)\|_T, \ldots, \|e_m(t)\|_T]^T$, with $\|r(t)\|_T$ defined similarly, we have

$$\|e(t)\|_T \leq \|r(t)\|_T + [d_{ij}\alpha_j]\|e(t)\|_T + [\mathrm{diag}(g_i\alpha_i)]\|e(t)\|_T,$$

or

$$A\|e(t)\|_T \leq \|r(t)\|_T$$

where A is the matrix defined in hypothesis (iv). Since matrix A is a Minkowski-matrix, that is, an M-matrix (see Section 2.3), it follows that A^{-1} exists and that $A^{-1} \geq 0$. Hence,

$$\|e(t)\|_T \leq A^{-1}\|r(t)\|_T, \qquad T, t\in R^+.$$

The proof of the theorem follows, letting $T \to \infty$ and assuming that $r_i \in S_\infty$, $i\in M$.

Proof of Theorem 3.2. For the ith Subsystem 3.4, $i\in M$, we have

$$Ey_i^2(n) = E\left[\sum_{\ell=0}^{n-1} w_i(n, \ell)f_i(\ell)e_i(\ell)\right]^2 = \sum_{\ell=0}^{n-1} w_i^2(n, \ell)\sigma_i^2(\ell)Ee_i^2(\ell)$$

$$\leq \sup_{0\leq\ell\leq n-1} Ee_i^2(\ell) \cdot \sum_{\ell=0}^{n-1} w_i^2(n, \ell)\sigma_i^2(\ell).$$

By the definitions of α_i and $\|\cdot\|_T$, we have

$$\left\|\sum_{\ell=0}^{n-1} w_i(n, \ell)f_i(\ell)e_i(\ell)\right\|_N = \|y_i(n)\|_N \leq \alpha_i\|e_i(n)\|_N \qquad N, n\in I^+.$$

It follows that

$$\|u_i(n)\|_N \le \|r_i(n)\|_N + \sum_{j=1}^{m} d_{ij}\|y_j(n)\|_N \le \|r_i(n)\|_N$$

$$+ \sum_{j=1}^{m} d_{ij}\alpha_j\|e_j(n)\|_N,$$

and the proof of this theorem is completed similarly as Theorem 3.1.

Proof of Theorem 3.3. If the ith Subsystem 3.2 is of Type 2 or 2S, we have, from the proof of Theorem 3.1,

$$\|e_i(t)\|_T \le \|u_i(t)\|_T + \alpha_i g_i\|e_i(t)\|_T,$$

where $[\int_0^t w_i^2(t, s)\sigma_i^2(s)ds]^{1/2} \le \alpha_i$, $t \in R^+$. If the ith Subsystem 3.2 is of Type 3, we have

$$e_i(t) = u_i(t) + \int_0^t w_i(t - s)e_i(s)d\beta_i(s)$$

or

$$e_i(t) = u_i(t) + f_{oi}\int_0^t w_i(t - s)e_i(s)ds$$

$$+ \int_0^t w_i(t - s)e_i(s)[d\beta_i(s) - f_{oi}ds].$$

Using a variation of constants technique for integral equations (see Miller [25, Chapter IV]), we obtain

$$e_i(t) = u_i(t) - \int_0^t w_i(t - s)e_i(s)[d\beta_i(s) - f_{oi}ds]$$

$$- \int_0^t \bar{r}_i(t - s)u_i(s)ds$$

$$+ \int_0^t \bar{r}_i(t - \tau) \int_0^\tau w_i(\tau - s)e_i(s)[d\beta_i(s) - f_{oi}ds]d\tau,$$

where $\bar{r}_i(t)$ denotes the resolvent of the kernel $f_{oi}w_i(t)$, and therefore satisfies

$$\bar{r}_i(t - s) = f_{oi}w_i(t - s) - \int_s^t f_{oi}w_i(t - s)\bar{r}_i(t - \tau)d\tau. \tag{A1}$$

It follows that

$$\|e_i(t)\|_T \leq \|u_i(t) - \int_0^t \bar{r}_i(t - s)u(s)ds\|_T$$

$$+ \|\int_0^t w_i(t - s)e_i(s)[d\beta_i(s) - f_{oi}ds]$$

$$- \int_0^t \int_\tau^t \bar{r}_i(t - \tau)w_i(\tau - s)e_i(s)d\tau[d\beta_i(s) - f_{oi}ds]\|_T,$$

where the order of integration of the iterated integral was changed. It now follows from Eq. A1 that

$$\|e_i(t)\|_T \leq \|u_i(t) - \int_0^t \bar{r}_i(t - s)u_i(s)ds\|_T$$

$$+ \|\frac{1}{f_{oi}} \int_0^t \bar{r}_i(t - s)e_i(s)[d\beta_i(s) - f_{oi}ds]\|_T .$$

The conditions of hypothesis (ii) of the theorem assure that $\bar{r}_i \in L_1(R^+) \cap L_2(R^+)$ and that

$$\tilde{r}_i(j\lambda) = \frac{f_{oi}\tilde{w}_i(j\lambda)}{1 + f_{oi}\tilde{w}_i(j\lambda)} ,$$

where $\tilde{r}_i$ and $\tilde{w}_i$ are the Fourier transforms of $\bar{r}_i$ and w_i, respectively. It now follows from Parseval's theorem that

$$E[\frac{1}{f_{oi}}\int_0^t \bar{r}_i(t - s)e_i(s)[d\beta_i(s) - f_{oi}ds]]^2$$

$$\leq \sup_{0\leq\tau\leq t} Ee_i^2(\tau)(\frac{1}{f_{oi}})^2 \int_0^\infty \bar{r}_i^2(t - s)\sigma_i^2 ds$$

$$= \sup_{0\leq\tau\leq t} Ee_i^2(\tau) \frac{\sigma_i^2}{2\pi}\int_{-\infty}^\infty \left|\frac{\tilde{w}_i(j\lambda)}{1 + f_{oi}\tilde{w}_i(j\lambda)}\right|^2 d\lambda.$$

We now have

$$\left\|\frac{1}{f_{oi}}\int_0^t \bar{r}_i(t - s)e_i(s)[d\beta_i(s) - f_{oi}ds]\right\|_T$$

$$\leq \|e_i(t)\|_T \cdot [\frac{\sigma_i^2}{2\pi}\int_{-\infty}^\infty \left|\frac{\tilde{w}_i(j\lambda)}{1 + f_{oi}\tilde{w}_i(j\lambda)}\right|^2 d\lambda] \leq \alpha_i\|e_i(t)\|_T,$$

where the definition of α_i for a Type 3 subsystem has been used (see condition (iv) of the theorem). Note that

$$\|u_i(t) - \int_0^t \bar{r}_i(t - s)u_i(s)ds\|_T \leq \|u_i(t)\|_T(1 + \int_0^\infty |\bar{r}_i(t)|dt)$$

$$\leq \gamma_i\|u_i(t)\|_T,$$

where the definition of γ_i for a Type 3 subsystem has been used (see condition (v) of the theorem). It now follows that

$$\|e_i(t)\|_T \leq \gamma_i \|u_i(t)\|_T + \alpha_i \|e_i(t)\|_T, \qquad i \in M.$$

Using the definition of $u_i(t)$ as given by Eq. 3.1, the definition of matrix A, as given in hypothesis (vi), and the definition of the vectors $\|e(t)\|_T$ and $\|r(t)\|_T$ as given in the proof of Theorem 3.1, we have

$$A\|e(t)\|_T \leq \mathrm{diag}[\gamma_i]\,\|r(t)\|_T.$$

The proof is now completed using an argument similar to the argument used to complete the proof of Theorem 3.1.

<u>Proof of Theorem 3.4</u>. From the proofs of Theorems 3.1 and 3.3 it follows that

$$\|e_i(t)\|_T \leq \gamma_i \|u_i(t)\|_T + \alpha_i \|e_i(t)\|_T,$$

where the parameters α_i and γ_i are appropriately defined for Subsystems 3.2 of Type 2, 2S, or 3. If the <u>ith</u> subsystem 3.2 is of Type 4, we have

$$e_i(t) = u_i(t) - \int_0^t w_i(t - s)e_i(s)d\beta_i(s) + \int_0^t h_i(t - s)e_i(s)ds.$$

Using the variation of constants technique employed in the proof of Theorem 3.3, we obtain

$$e_i(t) = u_i(t) - \int_0^t \bar{r}_i(t - s)u_i(s)ds - \int_0^t w_i(t - s)e_i(s)d\beta_i(s)$$
$$+ \int_0^t \bar{r}_i(t - \tau) \int_0^\tau w_i(t - s)e_i(s)d\beta_i(s)d\tau,$$

where $\bar{r}_i(t)$ denotes the resolvent of $h_i(t)$ (see Section 2.2). We have

$$\|e_i(t)\|_T \le \|u_i(t) - \int_0^t \bar{r}_i(t - s)u_i(s)ds\|_T$$
$$+ \|\int_0^t w_i(t - s)e_i(s)d\beta_i(s)$$
$$- \int_0^t \int_s^t \bar{r}_i(t - \tau)w_i(t - s)e_i(s)d\tau d\beta_i(s)\|_T$$
$$= \|u_i(t) - \int_0^t \bar{r}_i(t - s)u_i(s)ds\|_T$$
$$+ \|\int_0^t k_i(t - s)e_i(s)d\beta_i(s)\|_T,$$

where $k_i(t - s) = w_i(t - s) - \int_s^t \bar{r}_i(t - \tau)w_i(\tau - s)d\tau$. Hence

$$\tilde{k}_i(s) = \tilde{w}_i(s) - \tilde{w}_i(s)\tilde{r}_i(s) = \tilde{w}_i(s)[1 - \frac{\tilde{h}_i(s)}{1 + \tilde{h}_i(s)}] = \frac{\tilde{w}_i(s)}{1 + \tilde{h}_i(s)} .$$

Using Parsevals theorem, the definition of $\|\cdot\|_T$, and the definition of α_i for a Type 4 subsystem (see condition (v) of the theorem), we have

$$E\left[\int_0^t k_i(t-s)e_i(s)d\beta_i(s)\right]^2 \le \sup_{0\le s\le t} Ee_i^2(s)\cdot\int_0^\infty k_i^2(s)\sigma_i^2 ds$$

$$= \sup_{0\le s\le t} Ee_i^2(s)\left[\frac{\sigma_i^2}{2\pi}\int_{-\infty}^{\infty}|k_i(j\lambda)|^2 d\lambda\right]$$

$$= \sup_{0\le s\le t} Ee_i^2(s)\left[\frac{\sigma_i^2}{2\pi}\int_{-\infty}^{\infty}\left|\frac{\tilde{w}_i(j\lambda)}{1+\tilde{h}_i(j\lambda)}\right|^2 d\lambda\right],$$

and therefore

$$\left\|\int_0^t k_i(t-s)e_i(s)\beta_i(s)\right\|_T$$

$$\le \|e_i(t)\|_T\left[\frac{\sigma_i^2}{2\pi}\int_{-\infty}^{\infty}\left|\frac{\tilde{w}_i(j\lambda)}{1+\tilde{h}_i(j\lambda)}\right|^2 d\lambda\right]^{1/2}$$

$$\le \alpha_i\|e_i(t)\|_T.$$

As in the proof of Theorem 3.3, we have for a subsystem of Type 4

$$\left\|u_i(t)-\int_0^t \bar{r}_i(t-s)u_i(s)ds\right\|_T \le \gamma_i\|u_i(t)\|_T,$$

where $\gamma_i \ge 1 + \int_0^\infty |\bar{r}_i(t)|dt.$

If the ith Subsystem 3.2 is of Type 5, we have

$$e_i(t) = u_i(t) - \int_0^t h_i(t-s)e_i(s)ds,$$

so that

$$\|e_i(t)\|_T \leq \|u_i(t)\|_T + \|\int_0^t h_i(t - s)e_i(s)ds\|_T$$

and

$$\|\int_0^t h_i(t - s)e_i(s)ds\|_T \leq \|e_i(t)\|_T \int_0^\infty |h_i(t)|dt = \alpha_i\|e_i(t)\|_T,$$

where the definition of α_i for a Type 5 subsystem has been used. If the ith Subsystem 3.2 is of either Type 4 or 5, we have

$$\|e_i(t)\|_T \leq \gamma_i\|u_i\|_T + \alpha_i\|e_i(t)\|_T,$$

where $\gamma_i = 1$ if the ith Subsystem 3.2 is of Type 5 and $\gamma \geq 1 + \int_0^\infty |r_i(t)|dt$ if the ith subsystem is of Type 4. If the ith subsystem is of Type 5 we have $\alpha_i \geq \int_0^\infty |h_i(t)|dt$ and

$$\alpha_i \geq [\frac{1}{2\pi}\int_{-\infty}^\infty |\frac{\tilde{w}_i(j\lambda)}{1 + \tilde{h}_i(j\lambda)}|^2 d\lambda]^{1/2}$$

if the ith subsystem is of Type 4. Using the definition of $u_i(t)$ as given by Eq. 3.1, the definition of $\|e(t)\|$ and $\|r(t)\|_T$ as given in the proof of Theorem 3.1, and the definition of matrix A as given in hypothesis (vii) of the theorem, we obtain

$$A\|e(t)\|_T \leq \text{diag}[\gamma_i]\|r(t)\|_T$$

The proof is completed following an argument similar to that given in the proof of Theorem 3.1.

Proof of Theorem 3.5. Let H_k denote the operator which maps $u_k(t, \omega)$ into $e_k(t, \omega)$, so that $H_k u_k(t, \omega) = e_k(t, \omega)$. Note that H_k is a linear operator. In [40] it is shown that under hypothesis (i)-(v) of the theorem, there exists a process $r_k \in S_\infty$ such that $H_k r_k \in S_{\infty e} - S_\infty$. To demonstrate the instability of interconnected system 3.1, set $r_i(t) = 0$ $i \in M$, $i \neq k$, and as in [40], choose $r_k \in S_\infty$ such that $H_k r_k \in S_{\infty e} - S_\infty$. Since

$$u_k(t) = r_k(t) + \sum_{\substack{j=1 \\ j \neq k}}^{m} b_{kj} y_j(t)$$

we have

$$Ee_k^2(t) = E[H_k u_k(t)]^2 = E[H_k r_k(t) + \sum_{\substack{j=1 \\ j \neq k}}^{m} b_{kj} y_j(t)]^2$$

$$= E[H_k r_k(t)]^2 + \sum_{\substack{j=1 \\ j \neq k}}^{m} b_{kj}^2 E[H_k y_j(t)]^2$$

$$+ 2E[H_k r_k(t)][\sum_{\substack{j=1 \\ j \neq k}}^{m} b_{kj}(H_k y_j(t))]$$

$$+ 2E \sum_{\substack{j=1 \\ j \neq k}}^{m-1} \sum_{\substack{p=j+1 \\ p \neq k}}^{m} b_{kj} b_{kp} [H_k y_j(t)][H_k y_p(t)]. \qquad \text{(A2)}$$

In the following we will show that all crossproduct terms in Eq. A2 vanish, so that

$$Ee_k^2(t) = E[H_k r_k(t)]^2 + \sum_{\substack{j=1 \\ j \neq k}}^{m} b_{kj}^2 E[H_k y_j(t)]^2 \geq E[H_k r_k(t)]^2. \qquad \text{(A3)}$$

Because the supremum over $t \in R^+$ of $E[H_k r_k(t)]^2$ is unbounded (since $H_k r_k \in S_{\infty e} - S_\infty$), the conclusion of the theorem follows.

To show that the crossproduct terms in Eq. A2 vanish, consider first the term

$$2E[H_k r_k(t)][\sum_{\substack{j=1 \\ j\neq k}}^{m} b_{kj}(H_k y_j(t))]. \tag{A4}$$

Recall that

$$H_k r_k(t) = r_k(t) - \int_0^t w_k(t - s)[H_k r_k(s)]d\beta_k(s)$$

and

$$H_k y_j(t) = y_j(t) - \int_0^t w_k(t - s)[H_k y_j(s)]d\beta_k(s). \tag{A5}$$

Since by hypothesis $r_k(t)$ and $\beta_k(s)$ are stochastically independent, it follows from an argument involving conditional expectations and measurability concepts (see Arnold [1, Chapter 5]) that

$$Er_k(t) \int_0^t w_k(t - s)[H_k y_j(s)]d\beta_k(s) = 0.$$

Also $Ey_j(t)r_k(t) = 0$ for the same reason. Furthermore since $\beta_i(s)$ and $\beta_k(s)$ are stochastically independent, we have that

$$Ey_j(t) \int_0^t w_k(t - s)[H_k r_k(s)]d\beta_k(s)$$

$$= E \int_0^t w_j(t - s)e_j(s)d\beta_j(s)$$

$$\cdot \int_0^t w_k(t - s)[H_k r_k(s)]d\beta_k(s) = 0$$

(see Arnold [1, pp. 85]). Thus

$$E[H_k r_k(t)][H_k y_j(t)] = \sigma_k^2 \int_0^t w_k^2(t - s)E[H_k r_k(t)][H_k y_j(s)]ds.$$

The unique solution of this integral equation is $E[H_k r_k(t)][H_k y_j(t)] \equiv 0$. Hence the expression in Eq. A3 is identical to zero. Consider the expression

$$2E \sum_{\substack{j=1 \\ j\neq k}}^{m-1} \sum_{\substack{p=j+1 \\ p\neq k}}^{m} b_{kj} b_{kp} [H_k y_j(t)][H_k y_p(t)]. \tag{A6}$$

The term $H_k y_j(t)$ $(H_k y_p(t))$ is given by Eq. A5. Since $\dot{\beta}_j(t)$, $\dot{\beta}_k(t)$, and $\dot{\beta}_p(t)$ are mutually stochastically independent, we have in a fashion identical to the above,

$$E[H_k y_j(t)][H_k y_p(t)] = \sigma_k^2 \int_0^t w_k^2(t - s)E[H_k y_j(t)][H_k y_p(t)]dt \equiv 0.$$

Hence, the expression in Eq. A6 is zero. This completes the proof.

<u>Proof of Theorem 3.6</u>. Let H_k denote the operator on $s_{\infty e}$ which maps $u_k(n, \omega)$ into $e_k(n, \omega)$, so that $H_k u_k(n, \omega) = e_k(n, \omega)$. Note that H_k is a linear operator. In [40] it is shown that under hypotheses (i)-(v) of the theorem there exists a sequence $r_k \epsilon s_\infty$ such that $H_k r_k \epsilon s_{\infty e} - s_\infty$. To demonstrate the instability of interconnected system 3.3, let $r_i(n) = 0$, $i \epsilon M$, $i \neq k$, $n \epsilon I^+$, and, as in [40] choose, $r_k \epsilon s_\infty$ such that $H_k r_k \epsilon s_{\infty e} - s_\infty$. Since

$$u_k(n) = r_k(n) + \sum_{\substack{j=1 \\ j\neq k}}^{m} b_{ij}y_j(n)$$

we have, as in the proof of Theorem 3.5,

$$Ee_k^2(n) = E[H_k r_k(n)]^2 + \sum_{\substack{j=1 \\ j\neq k}}^{m} b_{kj}^2 E[H_k y_j(n)]^2$$

$$+ 2E[H_k r_k(n)][\sum_{\substack{j=1 \\ j\neq k}}^{m} b_{kj} H_k y_j(n)]$$

$$+ 2E \sum_{\substack{j=1 \\ j\neq k}}^{m-1} \sum_{\substack{p=j+1 \\ p\neq k}}^{m} b_{kj} b_{kp} [H_k y_j(n)][H_k y_p(n)]. \tag{A7}$$

As in the proof of Theorem 3.5, we will show that the crossproduct terms in Eq. A7 vanish so that

$$Ee_k^2(n) = E[H_k r_k(n)]^2 + \sum_{\substack{j=1 \\ j\neq k}}^{m} b_{kj}^2 E[H_k y_j(n)]^2 \geq E[H_k r_k(n)]^2. \tag{A8}$$

Since the supernum over $n \in I^+$ of $E[H_k r_k(n)]^2$ is unbounded (because $H_k r_k \in s_{\infty e} - s_\infty$), the conclusion of the theorem follows.

To show that the crossproduct terms in Eq. A7 vanish, consider first $E[H_k r_k(n)][H_k y_j(n)]$. Recall that

$$H_k r_k(n) = r_k(n) + \sum_{\ell=0}^{n-1} w_k(n-\ell) f_k(\ell)[H_k r_k(\ell)]$$

and

$$H_k y_j(n) = y_j(n) + \sum_{\ell=0}^{n-1} w_k(n-\ell) f_k(\ell)[H_k y_j(\ell)].$$

By hypothesis, $r_k(n)$ is stochastically independent of $f_j(p)$ n, $p \epsilon I^+$, so that

$$Er_k(n)y_j(n) = \sum_{\ell=0}^{n-1} w_j(n - \ell)Er_k(n)f_j(\ell)e_j(\ell) = 0.$$

Note that

$$Ey_j(n)f_k(\ell)[H_k r_k(\ell)] = \sum_{p=0}^{n-1} w_j(n - p)Ef_j(p)e_j(p)f_k(\ell)$$

$$\cdot [H_k r_k(\ell)] = 0,$$

since $f_i(p)$ and $f_k(\ell)$ are independent, by hypothesis. Also, note the stochastic independence of the pair $e_j(p)$, $f_j(p)$ and the pair $f_k(\ell)$ and $H_k r_k(\ell)$. It follows that

$$Ey_j(n) \sum_{\ell=0}^{n-1} w_k(n - \ell)f_k(\ell)[H_k r_k(\ell))] = 0.$$

Note further, due to the independence of $f_k(\ell)$ and $r_k(n) \cdot [H_k y_j(\ell)]$, that

$$Er_k(n)f_k(\ell)[H_k y_j(\ell)] = 0.$$

Hence

$$Er_k(n) \sum_{\ell=0}^{n-1} w_k(n - \ell)f_k(\ell)[H_k y_j(\ell)] = 0.$$

Therefore we have the recursive formula

$$E[H_k r_k(n)][H_k y_j(n)] = \sum_{\ell=0}^{n-1} w_k^2(r - \ell)\sigma_k^2 E[H_k r_k(\ell)][H_k y_j(\ell)],$$

$$n \epsilon I^+.$$

Now since

$$E[H_k r_k(0)][H_k y_j(0)] = r_k(0) y_j(0) = 0,$$

it follows that

$$E[H_k r_k(n)][H_k y_j(n)] = 0, \qquad n \in I^+. \tag{A9}$$

From a similar argument, it also follows that

$$E[H_k y_j(n)][H_k y_p(n)] = 0, \qquad n \in I^+. \tag{A10}$$

Equation A8 now follows from A7, A9 and A10, which completes the proof.

11. APPENDIX B. PROOFS OF RESULTS FROM CHAPTER 4

Proof of Theorem 4.1. From the assumption that $u(t, \omega)$ and $\beta(t)$ are independent, we have that

$$Eu(t, \omega)y(t, \omega) = Eu(t, \omega)\int_0^t g(t - \tau)\psi(e(\tau, \omega), \tau)d\beta(t) = 0.$$

From Eq. 4.1 and the above equation it follows that

$$Ee^2(t, \omega) = Eu^2(t, \omega) + Ey^2(t, \omega)$$

$$= Eu^2(t, \omega) + \sigma^2 \int_0^t g^2(t, \tau)E\psi^2(e(\tau, \omega), \tau)d\tau. \qquad \text{(B1)}$$

We "center" the nonlinearity by adding and subtracting terms:

$$Ee^2(t, \omega) = Eu^2(t, \omega) + \sigma^2 \int_0^t g^2(t - \tau)E[\psi^2(e(\tau, \omega), \tau)$$

$$- \rho Ee^2(\tau, \omega)]d\tau + \sigma^2\rho \int_0^t g^2(t - \tau)Ee^2(\tau, \omega)d\tau,$$

where $\rho = \frac{1}{2}(a^2 + b^2)$. Using operator notation, where

$$G_2x(t) = \int_0^t g^2(t - \tau)x(\tau)d\tau, \qquad t\epsilon R^+, \ x\epsilon L_{2e}(R^+),$$

and $Ix(t) = x(t)$, $x\epsilon L_{2e(1)}(R^+)$, we have

$$(I - \sigma^2\rho G_2)Ee^2(t, \omega) = Eu^2(t, \omega) + \sigma^2 G_2[E\psi^2(e(t, \omega), t)$$

$$- \rho Ee^2(t, \omega)].$$

By the classical Paley-Wiener result (see Miller [25, Chapter IV], or Holtzman [9, Chapter VIII]), condition (ii) of the theorem guarantees that $(I - \sigma^2 \rho G_2)$ is invertible on $L_2(R^+)$ and $L_{2e}(R^+)$. Furthermore $(I - \sigma^2 \rho G_2)^{-1}$ is a causal operator on these two spaces. Hence

$$Ee^2(t, \omega) = (I - \sigma^2 \rho G_2)^{-1} Eu^2(t, \omega) + \sigma^2 (I - \sigma^2 \rho G_2)^{-1} G_2 [E\psi^2(e(t, \omega), t) - \rho Ee^2(t, \omega)],$$

and truncating at T, $T \epsilon R^+$, we have

$$(Ee^2(t, \omega))_T = \pi_T (I - \sigma^2 \rho G_2)^{-1} (Eu^2(t, \omega))_T + \sigma^2 \pi_T (I - \sigma^2 \rho G_2)^{-1} \pi_T G_2 [E\psi^2(e(t, \omega), t) - \rho Ee^2(t, \omega)].$$

We now have

$$\|Ee^2(t, \omega)\|_T \leq \|\pi_T (I - \sigma^2 \rho G_2)^{-1}\| \cdot \|Eu^2(t, \omega)\|_T + \|\sigma^2 \pi_T (I - \sigma^2 \rho G_2)^{-1} \pi_2 G_2\| \cdot \|E\psi^2(e(t, \omega), t) - \rho Ee^2(t, \omega)\|_T. \tag{B2}$$

Since π_T is a projection on $L_2(R^+)$, it follows that

$$\|\pi_T (I - \sigma^2 \rho G_2)^{-1}\| \leq \|(I - \sigma^2 \rho G_2)^{-1}\|$$

and

$$\|\pi_T (I - \sigma^2 \rho G_2)^{-1} \pi_T G_2\| \leq \|(I - \sigma^2 \rho G_2)^{-1} G_2\|.$$

Note that

$$\frac{\sigma^2}{2} (b^2 - a^2) \|(I - \sigma^2 \rho G_2)^{-1} G_2\|$$

$$\leq \frac{\sigma^2}{2} (b^2 - a^2) \sup_{\lambda \epsilon R} |G_2(j\lambda)/(1 - \frac{\sigma^2}{2} (a^2 + b^2)G_2(j\lambda)| = \alpha < 1$$

by condition (iii) (see Holtzman [9, Chapter VIII]). For the nonlinear term, recall that $\rho = \frac{1}{2} (a^2 + b^2)$, and

$$\|E\psi^2(e(t, \omega, t) - \rho Ee^2(t, \omega)\|_T \leq \|b^2 Ee^2(t, \omega)$$
$$- \frac{1}{2} (a^2 + b^2) Ee^2(t, \omega)\|_T \leq \frac{1}{2} (b^2 - a^2) \|Ee^2(t, \omega)\|_T.$$

From Eq. B2 we may now write

$$\|Ee^2(t, \omega)\|_T \leq \|(I - \sigma^2 \rho G_2)^{-1} (Eu^2(t, \omega))\|_T + \alpha \|Ee^2(t, \omega)\|_T,$$

that is,

$$\|Ee^2(t, \omega)\|_T \leq \frac{1}{1 - \alpha} \|(I - \sigma^2 \rho G_2)^{-1} (Eu^2(t, \omega))\|_T .$$

Since by condition (iv) of the theorem $\{Eu^2(t, \omega)\} \epsilon L_2(R^+)$, we can let $T \to \infty$ and observe that $\{Ee^2(t, \omega)\} \epsilon L_2(R^+)$ as well. Since $E\psi^2(e(t, \omega), t) \leq b^2 Ee^2(t, \omega)$, it follows that $\{E\psi^2(e(t, \omega), t)\} \epsilon L_2(R^+)$ also. From Eq. B1 it may be seen that if we can show that

$$\int_0^t g^2(t - \tau) E\psi^2(e(\tau, \omega), \tau) d\tau \to 0 \text{ as } t \to \infty$$

then we are done, since condition (iv) of the theorem assures us that

$$Eu^2(t, \omega) \to 0 \text{ as } t \to \infty.$$

We use Lemma 2.2, which states that

$$\int_0^t k(t - \tau) k(\tau) d\tau \to 0 \text{ as } t \to \infty$$

provided $k \epsilon L_1(R^+) \cap L_2(R^+)$ and $h \epsilon L_2(R^+)$. Explicitly $g^2(t) = k(t)$ and $E\psi^2(e(t, \omega), t) = h(t)$. Condition (i) of the theorem guarantees that $k \epsilon L_1(R^+) \cap L_2(R^+)$ and we have shown above that $h \epsilon L_2(R^+)$. Hence

$$\int_0^t g^2(t - \tau)E\psi^2(e(\tau, \omega), \tau)d\tau \to 0 \text{ as } t \to \infty$$

and the proof is complete.

<u>Proof of Corollary 4.1</u>. By the principal of the argument (see Holtzman [9]), the condition

$$1 - \frac{\sigma^2}{2}(a^2 + b^2)G_2(s) \neq 0 \qquad \text{Re}(s) \geq 0$$

is satisfied if the locus of $1 - (\sigma^2/2)(a^2 + b^2)G_2(j\lambda)$, $\lambda \epsilon R$, does not encircle the point $(2(a^2 + b^2)/\sigma^2, 0)$. It can be seen that this point is always interior to the circle described in (ii) of the corollary. Requirement (iii) of the corollary is satisfied if

$$\frac{1}{2\sigma^2}(b^2 - a^2)|G_2(j\lambda)| < |1 - \frac{\sigma^2}{2}(a^2 + b^2)G_2(j\lambda)|, \qquad \lambda \epsilon R.$$

Defining $\xi = (\sigma^2/2)(a^2 + b^2)$, $\rho = (\sigma^2/2)(b^2 - a^2)$ and $z = G_2(j\lambda)$, condition (iii) is satisfied if, for $\lambda \epsilon R$,

$$\rho|z| < |1 - \xi z|$$

or if

$$\rho^2 zz^* < (1 - \xi z)(1 - \xi z^*)$$

or if

$$0 < \frac{1}{\xi^2 - \rho^2} \quad \frac{\xi}{\xi^2 - \rho^2} z - \frac{\xi}{\xi^2 - \rho^2} z^* + zz^*,$$

which is equivalent to

$$\left| z - \frac{\xi}{\xi^2 - \rho^2} \right| > \left(\frac{\xi}{\xi^2 - \rho^2}\right) - \left(\frac{1}{\xi^2 - \rho^2}\right).$$

Upon resubstitution for ξ, ρ and z, and simplification, it follows that

$$\left| G_2(j\lambda) - \frac{1}{2\sigma^2}\left(\frac{1}{a^2} - \frac{1}{b^2}\right) \right| > \frac{1}{2\sigma^2}\left(\frac{1}{a^2} - \frac{1}{b^2}\right), \qquad \lambda \epsilon R,$$

which is equivalent to the circle condition of requirement (ii) of the corollary.

12. APPENDIX C. PROOFS OF THEOREMS FROM CHAPTER 5

Proof of Theorem 5.1. Define the following subsets of Ω:

$$C_1(\psi_i) = \{\omega\epsilon\Omega: \text{ (i) and (ii) of Def. 2.3 are true for } \psi_i \epsilon \eta_{(N_i)}\};$$

$$C_2 = \{\omega\epsilon\Omega: \text{ (i), (iv), (v), (vi) of the theorem are satisfied for all } i\epsilon M\};$$

$$C_{3,i} = \{\omega\epsilon\Omega: \ e_i \epsilon E_{2(N_i)} \text{ satisfies Eq. 5.1)}\}; \text{ and}$$

$$C_{4,i} = \{\omega\epsilon\Omega: \ k_i \epsilon L_{1(N_i)}(R^+) \ \ L_{2(N_i)}(R^+)\}.$$

Define also

$$D = \{\bigcap_{i\epsilon M} C_1(\psi_i)\} \bigcap \{C_2\} \bigcap \{\bigcap_{i\epsilon M} C_{3,i}\} \bigcap \{\bigcap_{i\epsilon M} C_{4,i}\}.$$

Note that $P[D] = 1$. Using the definition of the operators K_i, Q_i, and π_T given in Section 5.2, we may rewrite Eq. 5.1 as

$$e_i(t, \omega) = u_i(t, \omega) - K_i Q_i e_i(t, \omega), \qquad t\epsilon R^+, \ \omega\epsilon D.$$

Truncating at $T\epsilon R^+$, we have

$$e_{i_T}(t, \omega) = u_{i_T}(t, \omega) - \pi_T K_i Q_i e_i(t, \omega), \qquad t, \ T\epsilon R^+, \ \omega\epsilon D.$$

Since K_i and Q_i are causal operators, $i\epsilon M$, we may write

$$e_{i_T}(t, \omega) = u_{i_T}(t, \omega) + \pi_T K_i \pi_T Q_i e_{i_T}(t, \omega), \qquad t, \ T\epsilon R^+, \ \omega\epsilon D$$

or

$$\pi_T(I + \frac{1}{2}(a_i + b_i)K_i)e_i(t, \omega) = u_i(t, \omega)$$

$$- \pi_T K_i \pi_T (Q_i - \frac{1}{2}(a_i + b_i)I)e_{i_T}(t, \omega)$$

where I denotes the identity operator on $L_{2(N_i)}(R^+, L_\infty(\Omega))$. For $\omega\epsilon D$, condition (iv) of the theorem assures that $(I + \frac{1}{2}(a_i + b_i)K_i)^{-1}$

exists on $L_{2(N_i)}(R^+)$ and is causal (see Sandberg [32] or Miller [25]), so that

$$e_{iT}(t, \omega) = \pi_T(Q_i + \frac{1}{2}(a_i + b_i)K_i)^{-1}u_{iT}(t, \omega)$$

$$- \pi_T(I + \frac{1}{2}(a_i + b_i)K_i)^{-1}$$

$$\pi_T K_i \pi_T(Q_i - \frac{1}{2}(a_i + b_i)I)e_{iT}(t, \omega) \qquad t, T\epsilon R^+, \omega\epsilon D, i\epsilon M,$$

and hence

$$\|e_{iT}\| \leq \|\pi_T(I + \frac{1}{2}(a_i + b_i)K_i)^{-1}u_{iT}\|$$

$$+ \|\pi_T(I + \frac{1}{2}(a_i + b_i)K_i)^{-1}\pi_T K_i\|$$

$$\cdot \|\pi_T(Q_i - \frac{1}{2}(a_i + b_i)I\| \cdot \|e_{iT}\|, \qquad \omega\epsilon D, T\epsilon R^+, i\epsilon M.$$

Since π_T is a projection on $L_{2(N_i)}(R^+)$ (for fixed $\omega\epsilon D$) we have

$$\|\pi_T(I + \frac{1}{2}(a_i + b_i)K_i)^{-1}\pi_T K_i\| \leq \|(I + \frac{1}{2}(a_i + b_i)K_i)^{-1}K_i\|$$

$$\leq \sup_{\lambda\epsilon R^+} [(I + \frac{1}{2}(a_i + b_i)\tilde{K}_i(j\lambda, \omega))^{-1}\tilde{K}_i(j\lambda, \omega)]$$

(see Sandberg [32]). Also note that

$$\|Q_i - \frac{1}{2}(a_i + b_i)\| \leq \frac{1}{2}(b_i - a_i) \qquad \omega\epsilon D, i\epsilon M.$$

Thus

$$\|\pi_T(I + \frac{1}{2}(a_i + b_i)K_i)^{-1}\pi_T K_i\| \cdot \|\pi_T(Q_i - \frac{1}{2}(a_i + b_i)I\| \leq \alpha_i$$

(by the definition of α_i given in (iii) of the theorem). In a similar fashion, by the definition of δ_i, we have

$$\|\pi_T(I + \frac{1}{2}\,(a_i + b_i)K_i)^{-1}\| \le \delta_i, \qquad \omega\epsilon D,\ i\epsilon M.$$

Hence

$$\|e_{iT}\| \le \|\pi_T(I + \frac{1}{2}\,(a_i + b_i)K_i)^{-1}u_{iT}\| + \alpha_i\|e_{iT}\|$$

$$\le \delta_i\|r_{iT}\| + \sum_{j=1}^{m} \|\pi_T(I + \frac{1}{2}\,(a_i + b_i)K_i)^{-1}\pi_T B_{ij}e_{jT}\|$$

$$+ \sum_{j=1}^{m} \|\pi_T(I + \frac{1}{2}\,(a_i + b_i)K_i)^{-1}\pi_T D_{ij}y_{jT}\| + \alpha_i\|e_{iT}\|$$

for $T\epsilon R^+$, $i\epsilon M$, and $\omega\epsilon D$. This implies that

$$\|e_{iT}\| \le \delta_i\|r_{iT}\| + \sum_{j=1}^{m} \|(I + \frac{1}{2}\,(a_i + b_i)K_i)^{-1}B_{ij}\|\cdot\|e_{jT}\|$$

$$+ \sum_{j=1}^{m} (I + \frac{1}{2}\,(a_i + b_i)K_i)^{-1}D_{ij}K_j\|\cdot\|Q_j\|\cdot\|e_{jT}\|$$

$$+ \alpha_i\|e_{iT}\|.$$

Using the fact that $\|Q_i\| \le \max\,(|b_i|, |a_i|)$ and the definitions of u_i, γ_{ij}, and ξ_{ij} as given in (iii) of the theorem, we have

$$\|e_{iT}\| \le \delta_i\|r_{iT}\| + \sum_{j=1}^{m} \gamma_{ij}\|e_{jT}\| + \sum_{j=1}^{m} \mu_j\xi_{jj}\|e_{jT}\| + \alpha_i\|e_{iT}\|.$$

Using vector notation with $\|e_T\| = [\|e_{iT}\|, \ldots, \|e_{mT}\|]^T$, and $\|r_T\|$ defined similarly, we have

$$A\|e_T\| \le [\mathrm{diag}(\delta_i)]\,\|r_T\|$$

by the definition of A given in (iii) of the theorem. Since by condition (v), A is an M-matrix, it possesses an inverse consisting of all non-negative elements, and hence

$$\|e_T\| \leq A^{-1}[\mathrm{diag}(\delta_i)]\|r_T\|.$$

For $\omega \epsilon D$ we have $r_i \epsilon L_{2(N_i)}(R^+)$ and by letting $T \rightarrow \infty$ we have $e_i \epsilon L_{2(N_i)}(R^+)$.

Using the matrix notation $[e] = [e_i(t, \omega), \ldots, e_m(t, \omega)]^T$, with $[r]$ and $[y]$ being defined similarly, and defining the operators on $L_{2(\Sigma N_i)}(R^+, L_\infty(\Omega))$

$$K = [\mathrm{diag}(K_i)]$$

$$Q = [\mathrm{diag}(Q_i)]$$

$$D = [D_{ij}]$$

and recalling the definitions of B_A and B_B from Section 5.2, we write Eq. 5.1 as

$$[e] = [r] - KQ[e] + B_A[e] + B_B[e] + DKQ[e]$$

By condition (vi) $(I - B_A)$ has a bounded inverse for $t \geq T^*$, $\omega \epsilon D$, for some $T^* \epsilon R^+$. We therefore have

$$[e] = (I - B_A)^{-1}\left\{[r] - KQ[e] + B_A[e] + D[y]\right\}.$$

Observe that since $e_i \epsilon L_{2(N_i)}(R^+)$ for $\omega \epsilon D$, then $Q_i e_i \epsilon L_{2(N_i)}(R^+)$ for $\omega \epsilon D$. Note that $(KQ[e] + B_A[e] + DKQ[E])$ may be written as a linear combination of integrals of the form $\int_0^t g_i(t - \tau, \omega)h_i(\tau, \omega)d\tau$, where for $\omega \epsilon D$, $g_i \epsilon K_{1(1\times 1)}(R^+) \cap K_{2(1\times 1)}(R^+)$ and $h_i \epsilon L_2(R^+)$. Therefore using Lemma 2.2 it follows that these integrals approach zero as $t \rightarrow \infty$. Since $|r_i| \rightarrow 0$ as $t \rightarrow \infty$ by hypothesis, the theorem follows.

Proof of Theorem 5.2. Define the following subsets of Ω:

$$C_1(\psi_i) = \{\omega \epsilon \Omega: \text{ (i) and (ii) of Def. 2.3 are true for } \psi_i \epsilon \eta_{(1)}\};$$

$$C_2 = \{\omega \epsilon \Omega: \text{ (v), (vi), (vii), and (viii) of the theorem are satisfied}\};$$

$$C_{3i} = \{\omega \epsilon \Omega: \ e_i(t, \omega) \text{ satisfies Eq. 5.1)}\};$$

$$C_{4i} = \{\omega \epsilon \Omega: \ k_i, \dot{k}_i \epsilon K_{1(1\times 1)}(R^+), \ k_i \epsilon K_{2(1\times 1)}(R^+)\}; \text{ and}$$

$$C_{5i} = \{\omega \epsilon \Omega: \ r_i, \dot{r}_i \epsilon L_{2(1)}(R^+), \ |r_i(t, \omega)| \to 0 \text{ as } t \to \infty\}.$$

Define the additional subset of Ω:

$$D = \{\bigcap_{i \epsilon M} C_1(\psi_i)\} \bigcap \{C_2\} \bigcap \{\bigcap_{i \epsilon M} C_{3i}\} \bigcap \{\bigcap_{i \epsilon M} C_{4i}\} \bigcap \{\bigcap_{i \epsilon M} C_{5i}\}.$$

Note that $P(D) = 1$. Condition (iv) guarantees that for ω D we may write the operator K_i as

$$K_i = K_{2i} K_{1i},$$

where K_{1i} is a linear mapping of E_2 into itself and K_{2i} maps E_2 into E_s characterized by its transform:

$$K_{2i}(j\lambda, \omega) = (1 + j\lambda q_i)^{-1}.$$

Also note that there exists a time-invariant linear mapping K_{i3} of E_s into E_2 such that

$K_{3i} K_{2i} = I$ (the identity operator on E_2); and

$K_{2i} K_{3i} = I$ (the identity operator on E_s).

Note that K_{i3} is characterized by the transform

$$K_{i3}(j\lambda,\ \omega) = (1 + j\lambda q_i).$$

We define the new variables $v_i(t,\ \omega) = K_{3i}e_i(t,\ \omega)$ (or $e_i(t,\ \omega) = K_{2i}v_i(t,\ \omega)$) and $z_i(t,\ \omega) = Q_iK_{2i}v_i(t,\ \omega)$. From Eq. 5.1 we have

$$K_{3i}e_i(t,\ \omega) = K_{3i}u_i(t,\ \omega) - K_{3i}K_{2i}K_{1i}Q_ie_i(t,\ \omega)$$

or

$$v_i(t,\ \omega) = K_{3i}u_i(t,\ \omega) - K_{1i}Q_iK_{2i}v_i(t,\ \omega)$$

or finally

$$v_i(t,\ \omega) = K_{3i}u_i(t,\ \omega) - K_{1i}z_i(t,\ \omega),$$

from which we may write

$$\begin{aligned} <(K_{3i}u_i)_T,\ z_{iT}> &= <(K_{1i}z_i)_T,\ z_{iT}> + <v_{iT},\ z_{iT}> \\ &= <(K_{1i}z_i)_T,\ z_{iT}> + <v_{iT},\ (Q_iK_{2i}v_i)_T>. \end{aligned} \tag{C1}$$

Since $Q_ix(t,\ \omega) \leq b_ix(t,\ \omega)$ for $x \epsilon L_2(R^+,\ L_\infty(\Omega))$, $\omega \epsilon D$ and $K_{2i}(j\lambda,\ \omega) = (1 + j\lambda q_i)^{-1}$, by an application of Lemma 2 of Zames [45], we have

$$<v_{iT},\ (Q_iK_{2i}v_i)_T> \geq b_i^{-1}\|z_{iT}\|^2, \quad T \epsilon R^+,\ v_i \epsilon E_{2(1)},\ \omega \epsilon D,\ i \epsilon M.$$

Also note that

$$\begin{aligned} <(K_{1i}z_i)_T,\ z_{iT}> &= <(K_{1i}z_{iT}),\ z_{iT}> \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{K}_{1i}(j\lambda,\ \omega)\tilde{z}_{iT}(j\lambda,\ \omega)\tilde{z}^*_{iT}(j\lambda,\ \omega)d\lambda \end{aligned}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{Re}\left\{(1 + j\lambda q_i)\tilde{K}_{1i}(j\lambda,\ \omega)\right\} |\tilde{z}_{iT}(j\lambda,\ \omega)|^2 d\lambda$$

$$\geq \frac{1}{2\pi} \int_{-\infty}^{\infty} (\delta_i - b_i^{-1}) |\tilde{z}_i(j\lambda,\ \omega)|^2 d\lambda$$

$$= (\delta_i - b_i^{-1}) \|z_{iT}\|^2 .$$

The above result uses condition (vi) of the theorem. Equation C1 becomes

$$b_i^{-1}\|z_{iT}\|^2 + (\delta_i - b_i^{-1})\|z_{iT}\|^2 \leq \langle (K_{3i}u_i)_T,\ z_{iT} \rangle$$

$$\leq \|K_{i3}u_{iT}\| \cdot \|z_{iT}\|$$

or

$$\|z_{iT}\| \leq \delta_i^{-1} \|K_{3i}u_{iT}\| .$$

We may now write

$$\|e_{iT}\| \leq \|(K_i z_i)_T\| + \|u_{iT}\| \leq \|K_i\| \delta_i^{-1} \|K_{3i}u_{iT}\| + \|u_{iT}\|$$

$$\leq \|K_i\| \delta_i^{-1} \left\{ \|K_{3i}r_{iT}\| + \sum_{j=1}^{m} \|K_{3i}B_{ij}e_{jT}\| \right\}$$

$$+ \|r_{iT}\| + \sum_{j=1}^{m} \|B_{ij}e_{jT}\| .$$

Using the notation $\alpha_i = \|K_i\|$, $\gamma_{ij} = \|K_{3i}B_{ij}\|$, $\beta_{ij} = \|B_{ij}\|$, we have

$$\|e_{iT}\| \leq \alpha_i \delta_i^{-1} \|K_{3i}r_{iT}\| + \alpha_i \delta_i^{-1} \sum_{j=1}^{m} (\gamma_{ij} + \beta_{ij}) \|e_{jT}\| + \|r_{iT}\|$$

or, using the vector notation used in the proof of Theorem 5.1, we have

$$A\|e_T\| \leq [\mathrm{diag}(\alpha_i \delta_i^{-1})]\|K_3 r_T\| + \|r_T\|,$$

where the matrix A is given in (vii) of the theorem. Since by hypothesis $r_i \epsilon L_2(R^+, L_\infty(\Omega))$, for $\omega \epsilon D$ we have $r_i \epsilon L_2(R^+)$ and hence $K_{3i} r_i \epsilon L_2(R^+)$ for $\omega \epsilon D$ (see Holtzman [9], Chapter VIII]). Since A is an M-matrix by condition (vii), we have as a result that $e_i \epsilon L_{2(1)}(R^+)$ for $\omega \epsilon D$, $i \epsilon M$. By the argument in the proof of Theorem 5.1, since $e_i \epsilon L_2(R^+)$ for $\omega \epsilon D$, then $Q_i e_i \epsilon L_2(R^+)$ for $\omega \epsilon D$ and $y_i \epsilon L_2(R^+)$ for $\omega \epsilon D$; and since $k_i \epsilon K_{1(1\times 1)}(R^+)$ $K_{2(1\times 1)}(R^+)$ for $\omega \epsilon D$, then by Lemma 2.2 and the fact that $|r_i(t, \omega)| \to 0$ as $t \to \infty$ we have $|e_i(t, \omega)| \to 0$ as $t \to \infty$ a.e.[P].

<u>Proof of Theorem 5.3</u>. From Eq. 5.5 we have

$$\frac{dx_i(t, \omega)}{dt} - A_i(\omega)x_i(t, \omega) = -\psi_i(x_i(t, \omega), t, \omega) + f_i(t, \omega) + \sum_{\substack{j=1 \\ j\neq i}}^{m} d_{ij}(\omega)x_j(t, \omega)$$

or, using the usual differential equation techniques,

$$\frac{d}{dt}[e^{-A_i(\omega)t} x_i(t, \omega)] = -e^{-A_i(\omega)t}\psi_i(x_i(t, \omega), t, \omega) + e^{-A_i(\omega)t} f_i(t, \omega) + \sum_{\substack{j=1 \\ j\neq i}}^{M} e^{-A_i(\omega)t} d_{ij}(\omega)x_j(t, \omega)$$

or

$$x_i(t, \omega) = e^{A_i(\omega)t}x_i(0) + \int_0^t e^{A_i(\omega)(t-\tau)}\psi_i(x_i(\tau, \omega), \tau, \omega)d\tau$$
$$+ \int_0^t e^{A_i(\omega)(t-\tau)}f_i(\tau, \omega)d\tau$$
$$+ \sum_{\substack{j=1 \\ j\neq i}}^{m} \int_0^t e^{A_i(\omega)(t-\tau)}d_{ij}(\omega)x_j(\tau, \omega)d\tau$$

which is of the form of Eq. 5.1 with the following assignments:

$$e_i(t, \omega) = x_i(t, \omega);$$

$$r_i(t, \omega) = e^{A_i(\omega)t}x_i(0) + \int_0^t e^{A_i(\omega)(t-\tau)}f_i(\tau, \omega)d\tau;$$

$$k_i(t - \tau, \omega) = e^{A_i(\omega)(t-\tau)};$$

$$B_{ij}e_j(t, \omega) = \int_0^t e^{A_i(\omega)(t-\tau)}d_{ij}(\omega)e_j(\tau, \omega)d\tau \quad i \neq j;$$

We will show that, under the conditions of this theorem, the hypothesis cf Theorem 5.1 is satisfied.

Note that the elements of $k_i(t, \omega)$ are linear combinations of $t^k\exp(\rho_i(\omega)t)$, $t^k\exp(\rho_i(\omega)t)\sin\sigma_j(\omega)t$, and $t^k\exp(\rho_j(\omega)t)\cos\sigma_j(\omega)t$, where $k\in\{0, 1, \ldots, N_i^{-1}\}$, $\rho_j(\omega)$ denotes the real part of the jth eigenvalue of $A_i(\omega)$ and $\sigma_j(\omega)$ is related to the jth eigenvalue of $A_i(\omega)$. Recall from the definition of a stochastically stable matrix that $\mathrm{Re}(\lambda_k(\omega)) \leq -\gamma < 0$ a.e.[P]. Hence we have $k_i \in K_{1(N_i\times N_i)}(R^+, L_\infty(\Omega))$, $i\in M$. Clearly $r_i \in L_{2(N_i)}(R^+, L_\infty(\Omega))$ by the same argument. Also

$$\int_0^t k_i(t - \tau, \omega) f_i(\tau, \omega) \to 0 \text{ as } t \to \infty \qquad \text{a.e.}[P]$$

by Lemma 2.2, since $k_i \in K_{1(N_i \times N_i)}(R^+)$ $K_{2(N_i \times N_i)}(R^+)$ for almost every ω and $f_i \in L_{2(N_i)}(R^+)$ for almost every ω. Obviously $|e^{A_i(\omega)} x_i(0)| \to 0$ as $t \to \infty$ a.e.[P]. Hence condition (ii) of Theorem 5.1 is satisfied. In this case

$$\det[I + \frac{1}{2}(a_i + b_i)\widetilde{K}_i(s, \omega)] = \det[I + \frac{1}{2}(a_i + b_i)(sI - A_i(\omega))^{-1}]$$

$$= \det[sI - A_i(\omega) + \frac{1}{2}(a_i + b_i)I] \cdot \det[(sI - A_i(\omega))^{-1}]$$

$$\neq 0, \ \mathrm{Re}(s) \geq 0 \qquad \text{a.e.}[P].$$

The above relation is due to conditions (i) and (iv) of Theorem 5.3. Conditions (iii) of Theorem 5.1 and condition (v) of Theorem 5.3 are equivalent as may be verified from

$$\|D_{ij}\| = 0 \qquad i, j \in M$$

and

$$\sup_{\lambda \in R^+} \Lambda[(I + \frac{1}{2}(a_i + b_i)\widetilde{K}_i(j\lambda, \omega))^{-1} K_i(j\lambda, \omega)]$$

$$= \sup_{\lambda \in R^+} \Lambda[(I + \frac{1}{2}(a_i + b_i)(j\lambda I + A_i(\omega))^{-1})^{-1}(j\lambda I + A_i(\omega))^{-1}]$$

$$= \sup_{\lambda \in R^+} \Lambda[(j\lambda I + A_i(\omega) + \frac{1}{2}(a_i + b_i)I)^{-1}],$$

and similarly

$$\sup_{\lambda \in R^+} \Lambda[(I + \frac{1}{2}(a_i + b_i)\widetilde{K}_i(j\lambda, \omega))^{-1}\widetilde{B}_{ik}(j\lambda, \omega)]$$

$$= \sup_{\lambda\epsilon R^+} \Lambda[I + \frac{1}{2}(a_i + b_i)(j\lambda I + A_i(\omega))^{-1})^{-1}(j\lambda I$$

$$+ A_i(\omega))^{-1} d_{ik}(\omega)]$$

$$= \sup_{\lambda\epsilon R^+} \Lambda[(j\lambda I + A_i(\omega) + \frac{1}{2}(a_i + b_i)I)^{-1} d_{ik}(\omega)].$$

Since B_{ij}, i, jϵM is a Type B operator, condition (iv) of Theorem 5.1 is satisfied. The proof is now complete.

Proof of Theorem 5.4. From Eq. 5.6 we have

$$\frac{dx_i(t, \omega)}{dt} - A_i(\omega)x_i(t, \omega) = v_i(\omega)\psi_i(\sigma_i(t, \omega), t, \omega) + f_i(t, \omega)$$

$$+ \sum_{\substack{j=1 \\ j\neq i}}^{m} d_{ij}(\omega)\sigma_j(t, \omega)$$

with

$$\sigma_i(t, \omega) = c_i^T(\omega)x_i(t, \omega).$$

As in the proof of Theorem 5.3 we arrive at

$$x_i(t, \omega) = e^{A_i(\omega)} x_i(0, \omega) + \int_0^t e^{A_i(\omega)(t-\tau)} v_i(\omega)\psi_i(\sigma_i(\tau, \omega)\tau, \omega)d\tau$$

$$+ \int_0^t e^{A_i(\omega)(t-\tau)} f_i(\tau, \omega)d\tau$$

$$+ \sum_{\substack{j=1 \\ j\neq i}}^{m} \int_0^t e^{A_i(\omega)(t-\tau)} d_{ij}(\omega)\sigma_j(\tau, \omega)d\tau.$$

Noting the definition of $\sigma_i(t, \omega)$, it is obvious that

$$\sigma_i(t, \omega) = c_i^T(\omega)e^{A_i(\omega)t}x_i(0, \omega)$$
$$+ c_i^T(\omega)\int_0^t e^{A_i(\omega)(t-\tau)}v_i(\omega)\psi_i(\sigma_i(\tau, \omega), \tau, \omega)d\tau$$
$$+ c_i^T(\omega)\int_0^t e^{A_i(\omega)(t-\tau)}f_i(\tau, \omega)d\tau$$
$$+ \sum_{\substack{j=1\\ j\neq i}}^{m} c_i^T(\omega)\int_0^t e^{A_i(\omega)(t-\tau)}d_{ij}(\omega)\sigma_i(\tau, \omega)d\tau.$$

This equation is of the form of Eq. 5.1 with the following identifications:

$$e_i(t, \omega) = \sigma_i(t, \omega)$$

$$r_i(t, \omega) = c_i^T(\omega)e^{A_i(\omega)t}x_i(0, \omega) + c_i^T(\omega)\int_0^t e^{A_i(\omega)(t-\tau)}f_i(\tau, \omega)d\tau$$

$$k_i(t, \omega) = c_i^T(\omega)e^{A_i(t)}v_i(\omega)$$

$$B_{ij}e_j(t, \omega) = \int_0^t c_i^T(\omega)e^{A_i(\omega)(t-\tau)}d_{ij}e_j(\tau, \omega)d\tau.$$

We will show that, under the conditions of Theorem 5.4, the hypothesis of Theorem 5.2 is satisfied. As in the proof of Theorem 5.3 we know that $e^{A_i(\omega)t} \in K_{1(N_i \times N_i)}(R^+, L_\infty(\Omega)) \cap K_{2(N_i \times N_i)}(R^+, L_\infty(\Omega))$. Since the elements of $C_i(\omega)$ and $v_i(\omega)$ are essentially bounded, we have that $k_i \in K_{1(1\times 1)}(R^+, L_\infty(\Omega)) \cap K_{2(i\times 1)}(R^+, L_\infty(\Omega))$, and $\dot{k}_i \in K_{2(1\times 1)}(R^+, L_\infty(\Omega))$. In a similar fashion $r_i, \dot{r}_i \in L_2(R^+, L_\infty(\Omega))$. The fact that

$$c_i^T(\omega)e^{A_i(\omega)t}x_i(0, \omega) \to 0 \text{ as } t \to \infty \qquad \text{a.e.}[P],\ i \in M$$

may be seen by recalling the constituents of $e^{A_i(\omega)t}$ as given in the proof of Theorem 5.3. Also, the term

$$c_i^T(\omega)\int_0^t e^{A_i(\omega)(t-\tau)} f_i(\tau, \omega)d\tau \rightarrow 0 \text{ as } t \rightarrow \infty \qquad \text{a.e.}[P],\ i\in M$$

by the same argument as in the proof of Theorem 5.3. Since

$$\tilde{K}_i(j\lambda, \omega) = c_i^T(\omega)(j\lambda I - A_i(\omega))^{-1}v_i(\omega)$$

it may be seen that condition (vi) of Theorem 5.2 is satisfied by condition (iv) of Theorem 5.4. Condition (vii) of Theorem 5.2 follows as a direct consequence of the form of $\tilde{K}_i(j\lambda, \omega)$ as given above and by condition (v) of Theorem 5.4.